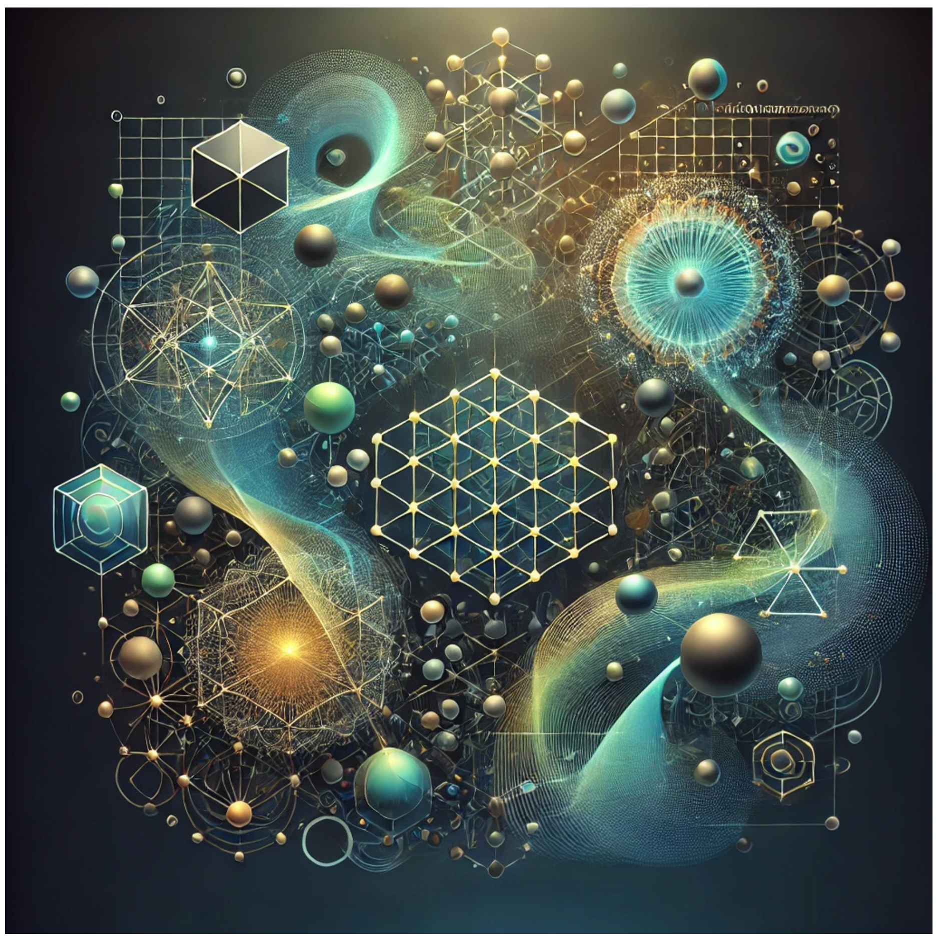
Takaaki Fujita
Florentin Smarandache

Advancing Uncertain Combinatorics

through Graphization, Hyperization, and Uncertainization:
Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Fourth Volume



HyperUncertain Set
(Collected Papers)

NSIA Publishing House

Takaaki Fujita
Florentin Smarandache

Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Fourth Volume

HyperUncertain Set

(Collected Papers)

This series explores the advancement of uncertain combinatorics through innovative methods such as graphization, hyperization, and uncertainization, incorporating concepts from fuzzy, neutrosophic, soft, and rough set theory, among others. Combinatorics and set theory are fundamental mathematical disciplines that focus on counting, arrangement, and the study of collections under specified rules. While combinatorics excels at solving problems involving uncertainty, set theory has expanded to include advanced concepts like fuzzy and neutrosophic sets, which are capable of modeling complex real-world uncertainties by accounting for truth, indeterminacy, and falsehood. These developments intersect with graph theory, leading to novel forms of uncertain sets in "graphized" structures, such as hypergraphs and superhypergraphs. Innovations like Neutrosophic Oversets, Undersets, and Offsets, as well as the Nonstandard Real Set, build upon traditional graph concepts, pushing the boundaries of theoretical and practical advancements. This synthesis of combinatorics, set theory, and graph theory provides a strong foundation for addressing the complexities and uncertainties present in mathematical and real-world systems, paving the way for future research and application.

Peer-reviewers:

Prof. **Lucian Tuțescu**
“Frații Buzești” National College, Craiova, Romania

Prof. Dr. **Mohamed Abdel-Basset**
Vice Dean for Community Service and Environmental Development
Faculty of Computers and Informatics, Zagazig University, Egypt

Prof. Dr. **Said Broumi**
Laboratory of Information Processing, Faculty of Science Ben M’Sik, University of Hassan II
Casablanca, Morocco

Takaaki Fujita, Florentin Smarandache

Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond

Fourth Volume

HyperUncertain Set

(Collected Papers)

NSIA Publishing House

Gallup - Guayaquil
United States of America - Ecuador
2025

Neutrosophic Science International Association
Publishing House

Division of Mathematics and Sciences
University of New Mexico
705 Gurley Ave., Gallup Campus
NM 87301, United States of America

University of Guayaquil
Av. Kennedy and Av. Delta
"Dr. Salvador Allende" University Campus
Guayaquil 090514, Ecuador

https://fs.unm.edu/NSIA/

ISBN 978-1-59973-817-8

Foreword

This book is the fourth volume in the series ***Collected Papers*** on ***Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond***.

This volume specifically delves into the concept of the ***HyperUncertain Se*t**, building on the foundational advancements introduced in previous volumes.

The series aims to explore the ongoing evolution of uncertain combinatorics through innovative methodologies such as graphization, hyperization, and uncertainization. These approaches integrate and extend core concepts from fuzzy, neutrosophic, soft, and rough set theories, providing robust frameworks to model and analyze the inherent complexity of real-world uncertainties.

At the heart of this series lies combinatorics and set theory—cornerstones of mathematics that address the study of counting, arrangements, and the relationships between collections under defined rules. Traditionally, combinatorics has excelled in solving problems involving uncertainty, while advancements in set theory have expanded its scope to include powerful constructs like fuzzy and neutrosophic sets. These advanced sets bring new dimensions to uncertainty modeling by capturing not just binary truth but also indeterminacy and falsity.

In this fourth volume, the integration of set theory with graph theory takes center stage, culminating in "graphized" structures such as hypergraphs and superhypergraphs. These structures, paired with innovations like Neutrosophic Oversets, Undersets, Offsets, and the Nonstandard Real Set, extend the boundaries of mathematical abstraction. This fusion of combinatorics, graph theory, and uncertain set theory creates a rich foundation for addressing the multidimensional and hierarchical uncertainties prevalent in both theoretical and applied domains.

The book is structured into thirteen chapters, each contributing unique perspectives and advancements in the realm of HyperUncertain Sets and their related frameworks:

The first chapter **(Advancing Traditional Set Theory with Hyperfuzzy, Hyperneutrosophic, and Hyperplithogenic Sets)** explores the evolution of classical set theory to better address the complexity and ambiguity of real-world phenomena. By introducing hierarchical structures like hyperstructures and superhyperstructures—created through iterative applications of power sets—it lays the groundwork for more abstract and adaptable mathematical tools. The focus is on extending three foundational frameworks: **Fuzzy Sets**, **Neutrosophic Sets**, and **Plithogenic Sets** into their hyperforms: **Hyperfuzzy Sets, Hyperneutrosophic Sets, and Hyperplithogenic Sets.**

These advanced concepts are applied across diverse fields such as statistics, clustering, evolutionary theory, topology, decision-making, probability, and language theory. The goal is to provide a robust platform for future research in this expanding area of study.

The second chapter **(Applications and Mathematical Properties of Hyperneutrosophic and SuperHyperneutrosophic Sets)** extends the work on Hyperfuzzy, Hyperneutrosophic, and Hyperplithogenic Sets by delving into their advanced applications and mathematical foundations. Building on prior research, it specifically examines Hyperneutrosophic and SuperHyperneutrosophic Sets, exploring their integration into: **Neutrosophic Logic**, **Cognitive Maps**,**Graph Neural Networks**, **Classifiers**, and **Triplet Groups.**

The chapter also investigates their mathematical properties and applicability in addressing uncertainties and complexities inherent in various domains. These insights aim to inspire innovative uses of hypergeneralized sets in modern theoretical and applied research.

The third chapter **(New Extensions of Hyperneutrosophic Sets – Bipolar, Pythagorean, Double-Valued, and Interval-Valued Sets)** studies advanced variations of Neutrosophic Sets, a mathematical framework defined by three membership functions: truth (T), indeterminacy (I), and falsity (F). By leveraging the concepts of Hyperneutrosophic and SuperHyperneutrosophic Sets, the study extends: **Bipolar Neutrosophic Sets**, **Interval-Valued Neutrosophic Sets**, **Pythagorean Neutrosophic Sets**, and **Double-Valued Neutrosophic Sets.**

These extensions address increasingly complex scenarios, and a brief analysis is provided to explore their potential applications and mathematical underpinnings.

Building on prior research, the fourth chapter (**Hyperneutrosophic Extensions of Complex, Single-Valued Triangular, Fermatean, and Linguistic Sets**) expands on Neutrosophic Set theory by incorporating recent advancements in Hyperneutrosophic and SuperHyperneutrosophic Sets. The study focuses on extending: **Complex Neutrosophic Sets**, **Single-Valued Triangular Neutrosophic Sets**, **Fermatean Neutrosophic Sets**, and **Linguistic Neutrosophic Sets.**

The analysis highlights the mathematical structures of these hyperextensions and explores their connections with existing set-theoretic concepts, offering new insights into managing uncertainty in multidimensional challenges.

The fifth chapter (**Advanced Extensions of Hyperneutrosophic Sets – Dynamic, Quadripartitioned, Pentapartitioned, Heptapartitioned, and m-Polar**) delves deeper into the evolution of Neutrosophic Sets by exploring advanced frameworks designed for even more intricate applications. New extensions include: **Dynamic Neutrosophic Sets**, **Quadripartitioned Neutrosophic Sets**, **Pentapartitioned Neutrosophic Sets**, **Heptapartitioned Neutrosophic Sets**, and **m-Polar Neutrosophic Sets.**

These developments build upon foundational research and aim to provide robust tools for addressing multidimensional and highly nuanced problems.

The sixth chapter (**Advanced Extensions of Hyperneutrosophic Sets – Cubic, Trapezoidal, q-Rung Orthopair, Overset, Underset, and Offset**) builds upon the Neutrosophic framework, which employs truth (T), indeterminacy (I), and falsity (F) to address uncertainty. Leveraging advancements in Hyperneutrosophic and SuperHyperneutrosophic Sets, the study extends: **Cubic Neutrosophic Sets**,

Trapezoidal Neutrosophic Sets, **q-Rung Orthopair Neutrosophic Sets**, **Neutrosophic Oversets**, **Neutrosophic Undersets**, and **Neutrosophic Offsets.**

The chapter provides a brief analysis of these new set types, exploring their properties and potential applications in solving multidimensional problems.

The seventh chapter (**Specialized Classes of Hyperneutrosophic Sets – Support, Paraconsistent, and Faillibilist Sets**) delves into unique classes of Neutrosophic Sets extended through Hyperneutrosophic and SuperHyperneutrosophic frameworks to tackle advanced theoretical challenges. The study introduces and extends: **Support Neutrosophic Sets**, **Neutrosophic Intuitionistic Sets**, **Neutrosophic Paraconsistent Sets**, **Neutrosophic Faillibilist Sets**, **Neutrosophic Paradoxist and Pseudo-Paradoxist Sets**, **Neutrosophic Tautological and Nihilist Sets**, **Neutrosophic Dialetheist Sets**, and **Neutrosophic Trivialist Sets.**

These extensions address highly nuanced aspects of uncertainty, further advancing the theoretical foundation of Neutrosophic mathematics.

The eight chapter (**MultiNeutrosophic Sets and Refined Neutrosophic Sets**) focuses on two advanced Neutrosophic frameworks: **MultiNeutrosophic Sets**, and **Refined Neutrosophic Sets.** Using Hyperneutrosophic and nn-SuperHyperneutrosophic Sets, these extensions are analyzed in detail, highlighting their adaptability to multidimensional and complex scenarios.

Examples and mathematical properties are provided to showcase their practical relevance and theoretical depth.

The ninth chapter (**Advanced Hyperneutrosophic Set Types – Type-m, Nonstationary, Subset-Valued, and Complex Refined**) explores extensions of the Neutrosophic framework, focusing on: **Type-m Neutrosophic Sets**, **Nonstationary Neutrosophic Sets**, **Subset-Valued Neutrosophic Sets**, and **Complex Refined Neutrosophic Sets.**

These extensions utilize the Hyperneutrosophic and SuperHyperneutrosophic frameworks to address advanced challenges in uncertainty management, expanding their mathematical scope and practical applications.

The tenth chapter (**Hyperfuzzy Hypersoft Sets and Hyperneutrosophic Hypersoft Sets**) integrates the principles of Fuzzy, Neutrosophic, and Soft Sets with hyperstructures to introduce: **Hyperfuzzy Hypersoft Sets**, and **Hyperneutrosophic Hypersoft Sets.**

These frameworks are designed to manage complex uncertainty through hierarchical structures based on power sets, with detailed analysis of their properties and theoretical potential.

The eleventh chapter (**A Review of SuperFuzzy, SuperNeutrosophic, and SuperPlithogenic Sets**) revisits and extends the study of advanced set concepts such as: **SuperFuzzy Sets**, **Super-Intuitionistic Fuzzy Sets**, **Super-Neutrosophic Sets**, and **SuperPlithogenic Sets**, including their specialized variants like quadripartitioned, pentapartitioned, and heptapartitioned forms.

The work serves as a consolidation of existing studies while highlighting potential directions for future research in hierarchical uncertainty modeling.

Focusing on decision-making under uncertainty, the tweve chapter (**Advanced SuperHypersoft and TreeSoft Sets**) introduces six novel concepts: **SuperHypersoft Rough Sets**,**SuperHypersoft Expert Sets**, **Bipolar SuperHypersoft Sets**, **TreeSoft Rough Sets**, **TreeSoft Expert Sets**, and **Bipolar TreeSoft Sets.**

Definitions, properties, and potential applications of these frameworks are explored to enhance the flexibility of soft set-based models.

The final chapter (**Hierarchical Uncertainty in Fuzzy, Neutrosophic, and Plithogenic Sets**) provides a comprehensive survey of hierarchical uncertainty frameworks, with a focus on Plithogenic Sets and their advanced extensions: **Hyperplithogenic Sets**, **SuperHyperplithogenic Sets.**

It examines relationships with other major concepts such as Intuitionistic Fuzzy Sets, Vague Sets, Picture Fuzzy Sets, Hesitant Fuzzy Sets, and multi-partitioned Neutrosophic Sets, consolidating their theoretical interconnections for modeling complex systems.

This volume not only reflects the dynamic interplay between theoretical rigor and practical application but also serves as a beacon for future research in uncertainty modeling, offering advanced tools to tackle the intricacies of modern challenges.

Takaaki Fujita, Florentin Smarandache

Table of Contents

Chapter 1

Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets I

Takaaki Fujita[1]*, Florentin Smarandache[2]

[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp

[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract. This work investigates the evolution of traditional set theory to address complex and ambiguous real-world phenomena. It introduces hierarchical hyperstructures and superhyperstructures, where superhyper-structures are formed by iteratively applying power sets to create nested abstractions.

The focus is placed on three foundational set-based frameworks—Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets—and their extensions into Hyperfuzzy Sets, HyperNeutrosophic Sets, and Hyperplithogenic Sets. These extensions are applied to various domains, including Statistics, TOPSIS, K-means Clustering, Evolutionary Theory, Topological Spaces, Decision Making, Probability, and Language Theory. By exploring these generalized forms, this paper seeks to guide and inspire further research and development in this rapidly expanding field.

Keywords: Fuzzy set, Neutrosophic set, Hyperstructure, Hyperfuzzy set, Hyperneutrosophic set

1. Introduction

1.1. *Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets*

Set theory, a foundational pillar of mathematics, provides a robust framework for analyzing collections of elements known as "sets" [118,192,210,237,415,416]. This paper focuses on three significant extensions of set theory—Fuzzy Sets [448], Neutrosophic Sets [354], and Plithogenic Sets [367]—and explores their further generalization into Hyperfuzzy [165], HyperNeutrosophic [146], and Hyperplithogenic Sets [146]. To establish context, we begin with an overview of these foundational concepts and their approaches to handling uncertainty.

Traditional set theory has evolved over time to address the inherent complexities and ambiguities of real-world scenarios. This evolution has given rise to a variety of advanced frameworks, including Fuzzy Sets [107, 407, 448–452, 465], Vague Sets [21, 86, 95, 196, 458], Hesitant Fuzzy Sets [409, 410], Soft Sets [32, 33, 147, 256, 268, 439], Rough Sets [289–295], Neutrosophic Offsets [361, 363, 379, 380], and Neutrosophic Sets [80, 124, 279, 351, 353, 354, 356, 359, 393, 419].

Takaaki Fujita and Florentin Smarandache, Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets

These methodologies each address specific dimensions of uncertainty. For instance, *Fuzzy Sets* assign membership values ranging from 0 to 1, allowing for partial belonging and providing a flexible framework for modeling uncertainty [448]. By contrast, *Neutrosophic Sets* extend this approach by introducing three degrees—truth, indeterminacy, and falsity—offering a more comprehensive toolset for capturing ambiguity in complex systems [354, 356].

A particularly noteworthy development is the concept of *Plithogenic Sets*, which generalize these frameworks further by incorporating multi-dimensional uncertainty and contradictions. This advanced model is especially useful for addressing challenges in analyzing complex systems where traditional methodologies may fall short [3, 170, 317, 344, 367, 368, 389, 394, 403].

Significant strides have been made in the development of these uncertain set theories, and their applications have expanded into related domains. For example, uncertain graph theories such as Fuzzy Graphs and Neutrosophic Graphs have emerged as essential tools for solving a wide array of theoretical and practical problems [12–14, 17–20, 22–24, 26, 81, 137, 145–147, 149, 153–155, 327].

1.2. *Hyperstructures and Superhyperstructures*

This subsection discusses the concepts of Hyperstructures and Superhyperstructures, which represent hierarchical structures in mathematics. A *Hyperstructure* is an extension of the power set concept applied to various mathematical frameworks [387, 388]. A *Superhyperstructure*, in turn, generalizes this idea by introducing the notion of n-th power sets, thereby creating a hierarchical and iterative abstraction. Superhyperstructures build upon hyperstructural principles, allowing for deeper abstraction and complexity [387, 388].

For example, in graph theory, a *Hypergraph* is a hyperstructure, while a *Superhypergraph* extends this concept by incorporating superedges and supervertices. Hypergraphs are generalized graphs where edges, termed hyperedges, can connect more than two vertices [60, 76, 119, 171–173]. Superhypergraphs provide an even broader generalization, enabling a higher level of abstraction and flexibility in graph-theoretic studies [138–140, 140, 141, 146, 148, 150, 151, 164, 187, 188, 314, 355, 370, 371, 377, 382, 385, 387].

The discussion above focuses primarily on superhypergraphs in graph theory. However, many other concepts have been explored in the realm of superhyperstructures, including superhyperalgebras [207, 208, 350, 375, 395], superhypertopology, superhyperrings [386], superhyperrough sets [146], superhyperdecision-making [143], superhypergraph neural networks [141], superhypergroups [224], superhyperfunctions [381, 385], superhyperweighted sets [146], superhypertopologies [383, 384, 395], superhyperlanguages [135, 144], PDCA superhypercycles [138], and superhypergames [143]. These extensions represent a wide array of mathematical and applied concepts being actively studied.

1.3. *Hyperfuzzy, HyperNeutrosophic, and Hyperplithogenic Sets*

This subsection provides an explanation of Hyperfuzzy, HyperNeutrosophic, and Hyperplithogenic Sets. Fuzzy Sets [146,165,214,399], Neutrosophic Sets [146], Plithogenic Sets [146], Soft Sets [1, 136, 154, 193, 206, 304, 338, 341, 366, 380], Rough Sets [146], and Vague Sets [146] have all been extended using Hyperstructures and n-SuperHyperstructures.

For example, in the case of Fuzzy Sets, these extensions are known as Hyperfuzzy Sets [134, 165, 214–217, 249, 253, 261, 277, 399] and SuperHyperfuzzy Sets [146, 364]. Similarly, for Neutrosophic Sets, the extensions include HyperNeutrosophic Sets [146, 364] and SuperHyperNeutrosophic Sets [146, 364], while for Plithogenic Sets, they include HyperPlithogenic Sets [146] and SuperHyperPlithogenic Sets [146]. This paper examines these concepts in detail. Readers interested in the underlying principles or further details about these concepts are encouraged to consult relevant literature, such as [146], as needed.

1.4. *Our Contribution in This Paper*

This subsection presents Our Contribution in This Paper. The HyperUncertain Set and SuperHyperUncertain Set, as extensions of the Uncertain Set, are believed to have wide-ranging applications; however, research on these concepts remains in its infancy.

In this paper, we explore the applicability of HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets to various domains, including Statistics, TOPSIS, K-means Clustering, Evolutionary Theory, Topological Spaces, Decision Making, Probability, and Language Theory.

It should be noted that this paper primarily focuses on theoretical mathematical considerations. Therefore, further experimental validation and application-oriented research will be necessary for practical implementation in specific technologies and fields.

Through these studies, we sincerely hope to contribute to the advancement of future research in this domain.

2. **Preliminaries and Definitions**

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. While we aim to present the core ideas, an exhaustive exploration of all terms is beyond the scope of this work. Readers interested in further details are encouraged to consult the cited references for additional insights.

2.1. *Fundamentals of Set Theory*

This subsection offers a concise overview of the basic principles of set theory. For a more detailed treatment, we recommend established references such as [195, 210, 213].

Definition 2.1 (Set)**.** [210] A *set* is a collection of distinct and clearly defined objects, called *elements*. For any object x, it is possible to determine whether x belongs to a given set. If x is an element of a set A, this is denoted as $x \in A$. Sets are commonly represented using curly braces. For example, the set $A = \{1, 2, 3\}$ contains the elements 1, 2, and 3.

Definition 2.2 (Subset)**.** [210] A set A is said to be a *subset* of another set B, written as $A \subseteq B$, if every element of A is also an element of B. This relationship can be expressed as:

$$A \subseteq B \iff \forall x\,(x \in A \implies x \in B).$$

If $A \subseteq B$ but $A \neq B$, A is referred to as a *proper subset* of B, denoted by $A \subset B$.

Definition 2.3 (Empty Set)**.** [210] The *empty set*, denoted by $\emptyset$, is the unique set that contains no elements. It is defined formally as:

$$\forall x\,(x \notin \emptyset).$$

For instance, the empty set can be represented as $\emptyset = \{\}$.

Definition 2.4 (Universal Set)**.** [210] The *universal set*, denoted by U, represents the set of all objects under consideration within a specific context. Any set A being analyzed is a subset of U. Formally:

$$A \subseteq U \quad \text{for any set } A.$$

Definition 2.5 (Operation)**.** [210] An *operation* is a function or rule that takes elements of a set S and produces another element within S. Formally, an operation $\circ$ on S is defined as:

$$\circ : S \times S \to S.$$

Examples include addition and multiplication, which are operations on the set of real numbers $\mathbb{R}$.

Definition 2.6 (Binary Operation)**.** [82] A *binary operation* on a set S is a function $* : S \times S \to S$ that combines two elements $a, b \in S$ to produce another element $a * b \in S$. For example, addition and subtraction are binary operations on $\mathbb{R}$.

2.2. *Hyperstructure and Superhyperstructure*

This subsection explores the concepts of Hyperstructure and Superhyperstructure, which serve as advanced mathematical frameworks for representing hierarchical systems. A *Hyperstructure* builds upon the notion of a powerset, providing a way to model relationships within sets. Extending this idea, a *Superhyperstructure* incorporates the n-th powerset, enabling the representation of multi-layered, hierarchical systems [135, 387, 388]. The formal definition of the n-th powerset is provided below.

Definition 2.7 (Base Set)**.** A *base set* is a fundamental set S from which more advanced structures, such as powersets and hyperstructures, are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within the specified domain}\}.$$

All elements in structures such as $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.8 (Powerset)**.** [141,325] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including both the empty set and S itself. Formally, it is defined as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.9 (n-th Powerset)**.** (cf. [141, 350, 387])

The n-th powerset of a set H, denoted by $P_n(H)$, is defined through an iterative process. Starting with the standard powerset, the construction proceeds as follows:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted by $P_n^*(H)$, is recursively defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set excluded.

Proposition 2.10. *(cf. [141, 350, 387])*

The n-th powerset generalizes the traditional concept of a powerset by applying the powerset operation iteratively.

Proof. The n-th powerset, $P_n(H)$, is constructed by repeatedly applying the standard powerset operation. Since $P_1(H) = P(H)$, and each subsequent step $P_{n+1}(H)$ builds upon the previous powerset, this iterative process inherently extends the original concept. Thus, the proposition follows directly from the definition. □

To provide a formal basis for the concepts of Hyperstructures and Superhyperstructures, we proceed with the following definitions and propositions.

Definition 2.11 (Classical Structure)**.** (cf. [350,387]) A *Classical Structure* is a mathematical construct defined on a non-empty set H, equipped with one or more *Classical Operations* that adhere to specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is an integer, and H^m denotes the m-fold Cartesian product of H. Examples include addition and multiplication, as found in common algebraic structures such as groups, rings, and fields.

Definition 2.12 (Hyperstructure)**.** (cf. [141, 350, 387]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is its powerset, and $\circ$ represents an operation defined on subsets within $\mathcal{P}(S)$.

Definition 2.13 (n-Superhyperstructure)**.** (cf. [350, 387]) An *n-Superhyperstructure* generalizes a Hyperstructure by utilizing the n-th powerset of a base set. It is formally represented as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ is an operation defined on elements of $\mathcal{P}_n(S)$.

Proposition 2.14. *An n-Superhyperstructure is inherently characterized by the properties of the n-th powerset.*

Proof. This follows directly from the definition of an n-Superhyperstructure, which is constructed explicitly using the n-th powerset $\mathcal{P}_n(S)$. □

Proposition 2.15. *A Hyperstructure is a special case of an n-Superhyperstructure for $n = 1$, and every n-Superhyperstructure generalizes the concept of a Hyperstructure.*

Proof. By definition, a Hyperstructure is based on the powerset $\mathcal{P}(S)$, which corresponds to the 1-st powerset $\mathcal{P}_1(S)$. When $n > 1$, the n-th powerset $\mathcal{P}_n(S)$ extends the structure, making the n-Superhyperstructure a generalization of the Hyperstructure. □

Additionally, we introduce the Dynamic n-SuperHyperStructure as an extension of the n-SuperHyperStructure framework. Although this paper does not provide a detailed discussion on the topic, it is worth noting that the concepts introduced later—such as the superhyperfuzzy set, superhyperneutrosophic set, and superhyperplithogenic set—can also be endowed with dynamic properties.

Definition 2.16 (Dynamic n-SuperHyperStructure)**.** Let S be a nonempty set, and let $\mathcal{P}_1(S) = \mathcal{P}(S)$ be the power set of S. For each integer $n \geq 2$, define:

$$\mathcal{P}_n(S) = \mathcal{P}(\mathcal{P}_{n-1}(S)),$$

the n-th power set of S.

A *Dynamic n-SuperHyperStructure* over a time domain T (which may be discrete or continuous, e.g., $T \subseteq \mathbb{R}$) is a triple:

$$\mathcal{DSH}_n = (S, \{\circ_t\}_{t \in T}, \Psi),$$

satisfying the following conditions:

(1) **Base Set and Hierarchy:** The structure is built upon the n-th powerset $\mathcal{P}_n(S)$. Each element of $\mathcal{P}_n(S)$ is a highly nested subset structure derived from S.

(2) **Time-Indexed Operation:** For each fixed $t \in T$, there is a binary operation:

$$\circ_t : \mathcal{P}_n(S) \times \mathcal{P}_n(S) \to \mathcal{P}_n(S).$$

The pair $(\mathcal{P}_n(S), \circ_t)$ is an n-SuperHyperStructure at time t. In other words, for each t, $\circ_t$ satisfies the structural axioms (if any are imposed) for n-SuperHyperStructures (e.g., closure under $\circ_t$, well-definedness of the operation, etc.).

(3) **Dynamic Evolution of Elements:** There exists an evolution function:

$$\Psi : T \times \mathcal{P}_n(S) \to \mathcal{P}_n(S),$$

which governs how elements of $\mathcal{P}_n(S)$ evolve over time. For instance, if $\{X_t\}_{t \in T}$ is a time-dependent family of elements in $\mathcal{P}_n(S)$, their evolution can be described by:

$$X_{t+\Delta t} = \Psi(t, X_t).$$

In a continuous-time setting, one might consider a limit $\Delta t \to 0$ and define a differential equation on $\mathcal{P}_n(S)$, but this requires additional structure.

(4) **Dynamic Evolution of the Operation:** The operation $\circ_t$ may also evolve with time. Formally, there is a function:

$$\Omega : T \times \mathcal{P}_n(S) \times \mathcal{P}_n(S) \to \mathcal{P}_n(S),$$

such that:

$$X \circ_t Y = \Omega(t, X, Y),$$

for all $X, Y \in \mathcal{P}_n(S)$. The choice of Ω defines how the binary operation changes as the system evolves over time.

(5) **Consistency and Well-Definedness:** For each $t \in T$, the structure $(\mathcal{P}_n(S), \circ_t)$ must be well-defined as an n-SuperHyperStructure. This means that $\circ_t$ takes two elements of $\mathcal{P}_n(S)$ and returns an element of $\mathcal{P}_n(S)$, and any additional axioms imposed on n-SuperHyperStructures (e.g., certain associative-like properties or closure properties) must hold for each fixed t.

- If Ψ is the identity function (i.e., $\Psi(t, X) = X$ for all t, X) and $\circ_t = \circ$ is time-invariant, the Dynamic n-SuperHyperStructure reduces to a static n-SuperHyperStructure.

- By appropriately choosing Ψ and Ω, one can model a wide variety of systems that exhibit hierarchical complexity and evolve over time, such as changing relational networks, adaptive classification schemes, or evolving hierarchical data structures.

2.3. *Fuzzy Set, Hyperfuzzy Set, and Superhyperfuzzy Set*

This subsection introduces the definitions of Fuzzy Set, Hyperfuzzy Set, and Superhyperfuzzy Set. Intuitively, these concepts extend the idea of fuzzy values into hierarchical structures, enabling a more nuanced representation of uncertainty. The formal definitions are provided below.

Definition 2.17. [448, 452] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 2.18 (Weather "Warmth")**.** Weather prediction involves forecasting atmospheric conditions, including temperature, precipitation, and wind, using scientific models, historical data, and computational methods (cf. [54, 323, 397]).

A fuzzy set τ models the degree of "warmth" in different temperature levels Y. Let:

$$Y = \{\text{Cool}, \text{Mild}, \text{Hot}\},$$

where the fuzzy set $\tau : Y \to [0, 1]$ assigns membership values to each element:

$$\tau(\text{Cool}) = 0.2, \quad \tau(\text{Mild}) = 0.6, \quad \tau(\text{Hot}) = 0.9.$$

These values represent the perceived degree of warmth for each temperature level.

Real-Life Application: A thermostat can use the fuzzy set τ to determine whether to increase cooling or heating based on the perceived warmth of the environment, rather than fixed thresholds.

Definition 2.19 (HyperFuzzy Set)**.** [72, 165, 214, 276, 399] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1])$ is called a *hyperfuzzy set* over X, where $\tilde{P}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Example 2.20 (Customer Satisfaction with Cars)**.** Customer satisfaction measures how well a product or service meets customer expectations, influencing loyalty, retention, and overall business success (cf. [56, 100, 228]).

A hyperfuzzy set $\tilde{\mu}$ represents satisfaction levels for different car attributes X:

$$X = \{\text{Price}, \text{Fuel Efficiency}, \text{Design}\}.$$

The hyperfuzzy set $\tilde{\mu} : X \to \tilde{P}([0,1])$ assigns subsets of satisfaction degrees to each attribute:

$$\tilde{\mu}(\text{Price}) = \{0.8, 0.9\}, \quad \tilde{\mu}(\text{Fuel Efficiency}) = \{0.6, 0.7, 0.9\}, \quad \tilde{\mu}(\text{Design}) = \{0.4, 0.6\}.$$

Each subset represents possible satisfaction levels due to uncertainty or varying preferences among customers.

Real-Life Application: Car manufacturers can use this model to analyze diverse customer opinions and develop vehicles that maximize satisfaction across multiple attributes.

Definition 2.21 (n-SuperHyperFuzzy Set)**.** [**?**, 146] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, representing the degrees of membership associated with A at the n-th level.

Example 2.22 (Example: Academic Performance in Subjects)**.** Academic performance evaluates a student's success in educational tasks, measured through grades, test scores, and overall learning achievements (cf. [298, 321]).

A superhyperfuzzy set $\tilde{\mu}_n$ models learning effectiveness across subject groups X:

$$X = \{\text{Math}, \text{Science}, \text{Art}\}.$$

Level 1: The first power set $\tilde{\mathcal{P}}_1(X)$ includes subsets of X:

$$\tilde{\mathcal{P}}_1(X) = \{\{\text{Math}\}, \{\text{Science}\}, \{\text{Math, Science}\}, \dots\}.$$

For a subset $A = \{\text{Math, Science}\}$, the first-level mapping assigns:

$$\tilde{\mu}_1(A) = \{0.7, 0.8\}.$$

Level 2: The second power set $\tilde{\mathcal{P}}_2(X)$ contains subsets of $\tilde{\mathcal{P}}_1(X)$. For example:

$$\tilde{\mathcal{P}}_2(X) = \{\{\{\text{Math}\}, \{\text{Science}\}\}, \{\{\text{Math, Science}\}\}, \dots\}.$$

For $B = \{\{\text{Math}\}, \{\text{Science}\}\}$, the second-level mapping assigns:

$$\tilde{\mu}_2(B) = \{0.6, 0.9\}.$$

Real-Life Application: Educational institutions can model and optimize the learning effectiveness of combined subject groups, designing curricula that cater to hierarchical dependencies between disciplines.

Theorem 2.23. *An n-superhyperfuzzy set generalizes both fuzzy sets and hyperfuzzy sets.*

Proof. This is evident. Refer to [146] if necessary. □

As demonstrated above, Fuzzy Sets can be extended to Hyperfuzzy Sets and SuperHyperfuzzy Sets, allowing related concepts to be similarly generalized. For instance, Bipolar Fuzzy Sets [27, 67, 68, 318, 459–461] can be extended to Bipolar Hyperfuzzy Sets and Bipolar SuperHyperfuzzy Sets; Tripolar Fuzzy Sets [308–311] to Tripolar Hyperfuzzy Sets and Tripolar SuperHyperfuzzy Sets; Neuro-Fuzzy Sets [11, 96, 128, 185, 197, 227, 267] to Neuro-Hyperfuzzy Sets and Neuro-SuperHyperfuzzy Sets; Hesitant Fuzzy SetsSets [324, 334, 409, 410, 437] to Hesitant Hyperfuzzy Sets and Hesitant SuperHyperfuzzy Sets; Picture Fuzzy Sets [103,104,159,251,335, 347, 398, 422, 426] to Picture Hyperfuzzy Sets and Picture SuperHyperfuzzy Sets; Pythagorean Fuzzy Sets [296, 427, 462] to Pythagorean Hyperfuzzy Sets and Pythagorean SuperHyperfuzzy Sets; Spherical Fuzzy Sets [34, 41, 130, 180, 181, 254, 262, 418] to Spherical Hyperfuzzy Sets and Spherical SuperHyperfuzzy Sets; and Nonstationary Fuzzy Sets [37, 161, 200, 201, 322, 443]to Nonstationary Hyperfuzzy Sets and Nonstationary SuperHyperfuzzy Sets. These extensions provide a framework for addressing increasingly complex and nuanced forms of uncertainty.

2.4. *Neutrosophic, HyperNeutrosophic, and SuperHyperNeutrosophic Sets*

Building on the concept of Fuzzy Sets, this subsection explores Neutrosophic, HyperNeutrosophic, and SuperHyperNeutrosophic Sets. Neutrosophic Sets extend Fuzzy Sets by incorporating the notion of indeterminacy, which captures situations that are neither true nor false [354]. This generalization provides a more robust framework for modeling real-world scenarios characterized by higher levels of uncertainty and complexity than Fuzzy Sets can address. Consequently, these concepts have been the focus of extensive research in numerous studies [156, 157, 225, 352, 356, 357, 373, 390–392, 396]. Relevant definitions are provided below.

Definition 2.24 (Neutrosophic Set)**.** [354] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 2.25 (A decision-making scenario in everyday life)**.** Decision-making is the process of evaluating options and selecting the best course of action to achieve specific goals or solve problems (cf. [28, 92, 297]).

Consider a decision-making scenario in everyday life: determining whether a given fruit (say, a banana) is "ripe" enough to eat. Let X be the set of all bananas in a grocery store. Define a Neutrosophic Set A representing the subset of bananas that are considered "ripe."

For each banana $x \in X$:

$T_A(x) \in [0, 1]$ measures the degree of certainty that the banana is ripe (e.g., based on color and smell).

$I_A(x) \in [0, 1]$ measures the indeterminacy or uncertainty about its ripeness (e.g., poor lighting, no direct smell test).

$F_A(x) \in [0, 1]$ measures the degree of certainty that the banana is not ripe (e.g., it is still green and too hard).

A possible assignment might be:

$$T_A(x) = 0.7, \quad I_A(x) = 0.2, \quad F_A(x) = 0.1,$$

indicating a reasonably high belief that the banana is ripe, some uncertainty due to ambiguous appearance, and a small possibility that it is not ripe.

Definition 2.26 (HyperNeutrosophic Set)**.** [146, 364] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0, 1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \le T + I + F \le 3.$$

Example 2.27. Now consider extending the scenario. Suppose there are multiple experts or tests evaluating the same banana's ripeness. One expert relies on color, another on smell, another on texture. Each produces a triplet (T, I, F) of evaluations. For each banana x, instead of assigning a single (T, I, F), we now have a *set* of such triplets, reflecting various perspectives.

For example:

$$\tilde{\mu}(x) = \{(0.8, 0.1, 0.1), (0.6, 0.3, 0.1), (0.75, 0.2, 0.05)\},$$

where each element in $\tilde{\mu}(x)$ might come from a different test:

- First triplet (0.8, 0.1, 0.1): Based on color analysis.
- Second triplet (0.6, 0.3, 0.1): Based on a smell test.
- Third triplet (0.75, 0.2, 0.05): Based on texture and firmness test.

Combining these different evaluations into one set for each banana gives a HyperNeutrosophic Set. Thus, each banana's "membership" in the concept of "ripe" is represented not just by one triple, but by a collection of triples capturing multiple dimensions of assessment.

Definition 2.28 (n-SuperHyperNeutrosophic Set)**.** [146, 364] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is similarly defined for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

Example 2.29. Consider a supply chain scenario (cf. [263, 400]) involving multiple stages:

(1) Farms where bananas are grown.
(2) Distribution centers where bananas from different farms are aggregated.
(3) Retailers who receive these bananas and sell them to consumers.

We can model the degrees of banana ripeness at different layers of this supply chain using Neutrosophic-type sets. If we let $n = 0$ correspond to a Neutrosophic Set, then at the farm stage (where $n = 0$), each banana is associated with a single (T, I, F)-triple indicating its ripeness:

$$\tilde{A}_0 : \tilde{\mathcal{P}}_0(X) \to \tilde{\mathcal{P}}_0([0,1]^3),$$

where $\tilde{\mathcal{P}}_0(X)$ is just X itself, and $\tilde{\mathcal{P}}_0([0,1]^3)$ represents assigning one (T, I, F)-triple per element.

At the distribution center stage (where $n = 1$), you aggregate the farm-level (Neutrosophic) evaluations into a HyperNeutrosophic Set. This means that for each banana, instead of having

just one (T, I, F)-triple, you now have a *set* of such triples (e.g., from different assessment criteria like color, smell, and texture):

$$\tilde{A}_1 : \tilde{\mathcal{P}}_1(X) \to \tilde{\mathcal{P}}_1([0,1]^3),$$

where $\tilde{\mathcal{P}}_1(X)$ indicates we are considering sets of elements with potentially multiple (T, I, F)-triplets assigned to each.

At the retail stage (where $n = 2$ or higher), you can form an n-SuperHyperNeutrosophic Set by integrating data from various distribution centers. This yields nested structures of sets of sets of (T, I, F)-triplets, capturing multiple layers of uncertainty and information aggregation:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where $\tilde{\mathcal{P}}_n(X)$ represents n-fold nested sets, and each nesting corresponds to an additional layer of aggregation.

For example, at a high n (several stages beyond just farms and distribution centers), you might have:

$$\tilde{A}_n(A) = \{\{\{\ldots(T_1, I_1, F_1), (T_2, I_2, F_2), \ldots\}\ldots\}\},$$

with multiple nested collections of (T, I, F)-triplets acquired from multiple levels of the supply chain. Each nesting step adds more complexity, reflecting a combination of many different assessments and conditions contributing to the final understanding of banana ripeness across the entire supply chain.

Theorem 2.30. *An n-superhyperneutrosophic set generalizes both neutrosophic sets and hyperneutrosophic sets.*

Proof. This is evident. Refer to [146] if necessary. □

Theorem 2.31. *An n-superhyperneutrosophic set generalizes n-superhyperfuzzy sets.*

Proof. This is evident. Refer to [146] if necessary. □

As demonstrated above, Neutrosophic Sets can be extended to Hyperneutrosophic Sets and SuperHyperneutrosophic Sets, enabling the generalization of related concepts. For example, Bipolar Neutrosophic Sets [112, 113, 413] can be extended to Bipolar Hyperneutrosophic Sets and Bipolar n-SuperHyperneutrosophic Sets. Similarly, Interval-Valued Neutrosophic Sets [332, 442, 447, 454, 456] can be generalized to Interval-Valued HyperNeutrosophic Sets and Interval-Valued n-SuperHyperneutrosophic Sets. Furthermore, Neutrosophic Soft Sets [223,

255, 333] can be extended to Hyperneutrosophic Soft Sets and n-SuperHyperneutrosophic Soft Sets.

Additionally, numerous types of Neutrosophic Sets are known, such as Hesitant Neutrosophic Sets [331, 463], Spherical Neutrosophic Sets [64, 169, 234, 326], and Intuitionistic Neutrosophic Sets [62, 63, 202]. These sets can similarly be extended, offering a broader framework for analyzing and applying Neutrosophic concepts across various fields.

2.5. *HyperPlithogenic Set*

The Plithogenic Set is known as a type of set that can generalize Neutrosophic Sets, Fuzzy Sets, and other similar sets [367, 368]. The definition of the Plithogenic Set is provided below.

Definition 2.32. [367, 368] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)*.
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

(1) *Reflexivity of Contradiction Function*:

$$pCF(a, a) = 0$$

(2) *Symmetry of Contradiction Function*:

$$pCF(a, b) = pCF(b, a)$$

Example 2.33. (cf. [142, 153]) The following examples of Plithogenic Sets are provided.

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Set*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Set*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Set*.

Definition 2.34 (HyperPlithogenic Set)**.** [146] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set* HPS over X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$

where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : (\bigcup_{i=1}^{n} Pv_i) \times (\bigcup_{i=1}^{n} Pv_i) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF).*

Definition 2.35 (n-SuperHyperPlithogenic Set)**.** [146] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^{n}, \{\tilde{pdf}_i^{(n)}\}_{i=1}^{n}, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,

  $$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$

  represents the k-th nested family of non-empty subsets of P_1.
- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^{n} P_{v_i} \times \bigcup_{i=1}^{n} P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  (1) Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^{n} P_{v_i}$,
  (2) Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^{n} P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Theorem 2.36. *An n-superhyperplithogenic set generalizes both plithogenic sets and hyperplithogenic sets.*

Proof. This is evident. Refer to [146] if necessary. □

Theorem 2.37. *An n-superhyperplithogenic set generalizes both n-superhyperfuzzy sets and n-superhyperneutrosophic sets.*

Proof. This is evident. Refer to [146] if necessary. □

2.6. *Plithogenic Sets: A Unified Framework for Handling Various Parameters*

Plithogenic Sets provide a unified framework for generalizing a wide range of concepts, as described below. This capability extends naturally to HyperPlithogenic Sets and n-SuperHyperPlithogenic Sets, maintaining the same principles of generalization.

Example 2.38. (cf. [142, 153]) The following table provides concrete examples of Plithogenic Sets and their transformations, highlighting the number of parameters (s) and their associated features. These examples also illustrate the diversity and complexity of Plithogenic Sets.

It is important to emphasize that while the number of parameters is relevant for classification, their names and meanings must also be considered, particularly when analyzing applications. For example, the Quadripartitioned Neutrosophic Set is known to be more practical and applicable compared to Turiyam Neutrosophic Sets or Ambiguous Sets [351]. Additionally, it should be noted that other extended sets, such as the Multi-fuzzy set [29, 35, 339, 340, 440] and the Refined Neutrosophic Set [111, 220, 275, 313, 412, 414], also exist. These sets include multiple parameters representing the same type of uncertainty, further enriching their descriptive capabilities.

s	t	*Type of Plithogenic Set*
1	1	Fuzzy Set [448]
2	1	Intuitionistic Fuzzy Set [46–53]
		Vague Set [86, 95, 196]
		Paraconsistent Set [87, 131, 247, 424, 425]
3	1	Neutrosophic Set [353, 354]
		Hesitant Fuzzy Set [409, 410]
		Three-way Fuzzy Set [199, 420]
		Spherical Fuzzy Set [24, 179, 181]
4	1	Quadripartitioned Neutrosophic Set [71, 90]
		Double-Valued Neutrosophic Set [120, 219, 230, 252, 428]
		Ambiguous Set [345, 346, 351]
		Turiyam Neutrosophic Set [158, 343]
5	1	Pentapartitioned Neutrosophic Set [106]
6	1	Hexapartitioned Neutrosophic Set [288]
7	1	Heptapartitioned Neutrosophic Set [78, 271]
8	1	Octapartitioned Neutrosophic Set
9	1	Nonapartitioned Neutrosophic Set

TABLE 1. Examples of Plithogenic Sets based on parameters s and t.

3. **Result: Application of HyperNeutrosophic Sets to Various Sciences**

This paper explores the application of HyperNeutrosophic concepts across various scientific fields. It is worth noting that if a HyperNeutrosophic Set can be applied to a particular domain, it is natural to consider that Hyperfuzzy Sets and Hyperplithogenic Sets could also find application in similar contexts. Furthermore, for SuperHyperNeutrosophic Sets, it is equally plausible to examine the applicability of SuperHyperfuzzy Sets and SuperHyperplithogenic Sets within the same or related domains.

3.1. *HyperNeutrosophic Probability*

Probability quantifies the likelihood of an event occurring, represented as a value between 0 (impossible) and 1 (certain) [110, 264, 284, 285, 299, 306]. Related concepts, such as Hyper-Probability and n-SuperHyperProbability, have also been explored [85, 144].

Furthermore, extended frameworks like Fuzzy Probability [84, 245, 300, 330, 457, 464], Neutrosophic Probability [358, 362], and Plithogenic Probability [365] are well-established. These concepts will be analyzed in the following sections.

Definition 3.1 (Neutrosophic Probability)**.** [358, 362] Let Ω be a sample space, and $\mathcal{F}$ be a σ-algebra of subsets of Ω (events). A *Neutrosophic Probability (NP)* is a function:

$$NP : \mathcal{F} \to [0,1]^3,$$

where for each event $A \in \mathcal{F}$, $NP(A) = (T(A), I(A), F(A))$ satisfies:

$$0 \leq T(A), I(A), F(A) \leq 1 \quad \text{and} \quad 0 \leq T(A) + I(A) + F(A) \leq 3.$$

Here:

- $T(A)$: Truth degree of A (likelihood of A being true),
- $I(A)$: Indeterminacy degree of A (uncertainty about A),
- $F(A)$: Falsity degree of A (likelihood of A being false).

Example 3.2 (Applications of Neutrosophic Probability)**.** This section provides two precise examples of Neutrosophic Probability and their applications in real-life contexts.

1. Weather Forecasting: Consider $\Omega = \{\text{rain, sunny, cloudy}\}$, and let the event $A = \{\text{rain}\}$ represent the occurrence of rain. A Neutrosophic Probability function assigns:

$$NP(A) = (0.75, 0.15, 0.10),$$

where:

- $T(A) = 0.75$: The degree of truth, representing the probability of rain based on meteorological predictions and atmospheric data.

- $I(A) = 0.15$: The indeterminacy, accounting for uncertainty due to limited data or conflicting weather models.
- $F(A) = 0.10$: The degree of falsity, indicating the probability that rain will not occur based on historical weather patterns.

This enables a nuanced understanding of the likelihood of rain, integrating truth, uncertainty, and contradiction.

2. Medical Diagnosis: In a diagnostic scenario, consider $\Omega = \{\text{healthy, sick}\}$ with $A = \{\text{sick}\}$ representing a positive diagnosis for an illness. A Neutrosophic Probability function assigns:

$$NP(A) = (0.80, 0.10, 0.10),$$

where:

- $T(A) = 0.80$: The probability of being sick based on test results.
- $I(A) = 0.10$: Indeterminacy arising from inconclusive or incomplete diagnostic information.
- $F(A) = 0.10$: The probability of being healthy despite the test results, indicating potential errors or anomalies in testing.

This provides a comprehensive representation of diagnostic uncertainty, enhancing clinical decision-making.

Definition 3.3 (HyperNeutrosophic Probability)**.** Let Ω be a sample space, and $\mathcal{F}$ be a σ-algebra of subsets of Ω. A *HyperNeutrosophic Probability (HNP)* is a function:

$$HNP : \mathcal{F} \to \mathcal{P}([0, 1]^3),$$

where $\mathcal{P}([0, 1]^3)$ is the power set of the neutrosophic unit cube $[0, 1]^3$. For each event $A \in \mathcal{F}$, $HNP(A)$ is defined as:

$$HNP(A) = \{(T_k(A), I_k(A), F_k(A)) \mid k \in \mathcal{K}_A\},$$

where:

- $T_k(A), I_k(A), F_k(A) \in [0, 1]$ are the k-th truth, indeterminacy, and falsity degrees of A,
- $\mathcal{K}_A$ is the index set of evaluations for A,
- Each triple satisfies:

$$0 \leq T_k(A) + I_k(A) + F_k(A) \leq 3.$$

Example 3.4 (Applications of HyperNeutrosophic Probability)**.** This section elaborates on two concrete examples of HyperNeutrosophic Probability, illustrating its utility in real-world scenarios.

1. Financial Risk Assessment (cf. [236]): In assessing credit default risks, let $\Omega = \{\text{default, no default}\}$ with $A = \{\text{default}\}$ representing the event of a credit default. A HyperNeutrosophic Probability function assigns:

$$HNP(A) = \{(0.85, 0.10, 0.05), (0.75, 0.15, 0.10), (0.80, 0.12, 0.08)\},$$

where:

- Each triple $(T_k(A), I_k(A), F_k(A))$ corresponds to evaluations from different financial models or analysts.
- $T_k(A)$: Reflects the degree of truth (likelihood of default based on financial metrics).
- $I_k(A)$: Captures uncertainty due to varying assumptions or incomplete data.
- $F_k(A)$: Represents the likelihood of no default, considering optimistic scenarios or mitigating factors.

This enables risk analysis by consolidating diverse expert perspectives.

2. Environmental Risk Analysis(cf. [178]): In evaluating the impact of industrial activities, let $\Omega = \{\text{low impact, medium impact, high impact}\}$ with $A = \{\text{high impact}\}$ representing a severe environmental impact. A HyperNeutrosophic Probability function assigns:

$$HNP(A) = \{(0.65, 0.20, 0.15), (0.70, 0.15, 0.15), (0.60, 0.25, 0.15)\},$$

where:

- Each triple corresponds to evaluations from different environmental studies or simulation models.
- $T_k(A)$: Degree of truth based on environmental assessments or emissions data.
- $I_k(A)$: Indeterminacy arising from uncertain or conflicting data.
- $F_k(A)$: Degree of falsity representing the likelihood of minimal or no impact.

This facilitates a robust decision-making process by integrating multiple environmental perspectives and addressing uncertainties.

Definition 3.5 (n-SuperHyperNeutrosophic Probability)**.** Let Ω be a sample space, and $\mathcal{F}$ be a σ-algebra of subsets of Ω. An *n-SuperHyperNeutrosophic Probability (n-SHNP)* is a function:

$$HNP^{(n)} : \mathcal{F} \to \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ is the n-th nested power set of $[0,1]^3$. For each event $A \in \mathcal{F}$, $HNP^{(n)}(A)$ is defined recursively as:

$$HNP^{(n)}(A) = \begin{cases} [0,1]^3, & \text{if } n = 0, \\ \mathcal{P}(HNP^{(n-1)}(A)), & \text{if } n \geq 1. \end{cases}$$

This structure captures higher-order uncertainties and allows for multi-layered probability evaluations.

Theorem 3.6 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Probability possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.7 (Reduction to Neutrosophic Probability)**.** *For $n = 0$, n-SuperHyperNeutrosophic Probability (n-SHNP) reduces to standard Neutrosophic Probability (NP).*

Proof. By definition, when $n = 0$, we have:

$$HNP^{(0)}(A) = [0, 1]^3,$$

where $[0, 1]^3$ represents a single neutrosophic triple $(T(A), I(A), F(A))$ with:

$$0 \leq T(A), I(A), F(A) \leq 1 \quad \text{and} \quad 0 \leq T(A) + I(A) + F(A) \leq 3.$$

This is precisely the definition of a Neutrosophic Probability for the event A. Therefore, 0-SHNP is equivalent to NP. □

Theorem 3.8 (Hierarchical Complexity)**.** *For every $n \geq 1$, n-SHNP is strictly more complex, in terms of set structure, than $(n-1)$-SHNP.*

Proof. Consider the recursive definition:

$$HNP^{(n)}(A) = \mathcal{P}(HNP^{(n-1)}(A)).$$

When moving from $(n-1)$ to n, we apply the power set operation $\mathcal{P}$, which generates the set of all subsets of $HNP^{(n-1)}(A)$. The power set operation always produces a strictly larger and more complex structure for non-empty sets. Hence, $HNP^{(n)}(A)$ introduces an additional layer of combinatorial and hierarchical complexity compared to $HNP^{(n-1)}(A)$, making n-SHNP strictly more complex than $(n-1)$-SHNP. □

Theorem 3.9 (Consistency of Truth, Indeterminacy, and Falsity)**.** *For any event $A \in \mathcal{F}$, the total measure of truth, indeterminacy, and falsity at each level n satisfies:*

$$0 \leq \sum_{(T,I,F) \in HNP^{(n)}(A)} \big(T + I + F\big) \leq 3 \cdot |HNP^{(n)}(A)|.$$

Proof. By definition, each triple $(T_k, I_k, F_k) \in HNP^{(n)}(A)$ satisfies:

$$0 \leq T_k + I_k + F_k \leq 3.$$

For the n-th level, $HNP^{(n)}(A)$ consists of subsets of $(n-1)$-SHNP structures due to the power set operation. The total measure over all elements in $HNP^{(n)}(A)$ is given by:

$$\sum_{(T,I,F)\in HNP^{(n)}(A)} \big(T+I+F\big),$$

which must be bounded above by $3 \cdot |HNP^{(n)}(A)|$ due to the constraints on each triple. The lower bound 0 is achieved if all measures are zero. This consistency holds at all levels n. □

Theorem 3.10 (Projection Property)**.** *For any $n \geq 1$, there exists a projection function:*

$$\pi_n : HNP^{(n)}(A) \to HNP^{(n-1)}(A),$$

which maps an n-SHNP structure to its corresponding $(n-1)$-SHNP structure without violating the probability constraints.

Proof. Define π_n as the inverse of the power set operation:

$$\pi_n(S) = \{x \in S \mid x \in HNP^{(n-1)}(A)\}.$$

By construction, $HNP^{(n)}(A) = \mathcal{P}(HNP^{(n-1)}(A))$, so every element of $HNP^{(n)}(A)$ is a subset of $HNP^{(n-1)}(A)$. Applying π_n preserves the original structure of $HNP^{(n-1)}(A)$. The constraints on truth, indeterminacy, and falsity are inherited directly from the $(n-1)$-level, ensuring that π_n maintains consistency with the probability framework. □

Theorem 3.11 (Continuity under Nested Refinement)**.** *If the probability values (T, I, F) are adjusted within the bounds of the constraints at level n, the resulting n-SHNP structure varies continuously with respect to those adjustments.*

Proof. Each level n of the n-SHNP structure is defined recursively via the power set operation. Small changes in the neutrosophic values (T, I, F) at any level propagate through the hierarchy as adjustments to the subsets in subsequent levels. Since the power set operation involves discrete combinations of subsets, the adjustments in the probability values lead to corresponding changes in the elements of the higher-level power sets without discontinuities. Thus, the structure of $HNP^{(n)}(A)$ varies continuously under bounded perturbations of the probability values at any level. □

Similarly, HyperFuzzy Probability, HyperPlithogenic Probability, SuperHyperFuzzy Probability, and SuperHyperPlithogenic Probability can also be defined.

This subsection introduces HyperNeutrosophic Statistics. Statistics is the discipline concerned with the collection, organization, analysis, and interpretation of data, aiding in making informed decisions and predictions [74, 75, 429]. Statistics is deeply connected to probability theory and encompasses important concepts such as random variables [284, 299], expectations [61, 243], and distribution functions [121, 246].

Extensions of classical statistics, such as Fuzzy Statistics [83,115,221,239,278,307,408,423], Neutrosophic Statistics [42–45, 93, 360], and Plithogenic Statistics [369, 374, 378], have been developed to address various forms of uncertainty. This paper further extends these concepts to HyperNeutrosophic Statistics and SuperHyperNeutrosophic Statistics. Relevant definitions are provided below.

Definition 3.12 (Neutrosophic Statistics)**.** [93, 360] Neutrosophic Statistics (NS) is a generalization of classical statistics and interval statistics, developed to handle data characterized by indeterminacy, vagueness, and conflicting information. It relies on Neutrosophic Logic, Probability, and Set Theory to provide a robust framework for analyzing uncertain datasets. Components of Neutrosophic Statistics.

(1) *Neutrosophic Probability Distribution (NPD):* For a random event x, its occurrence is described by a triple:

$$NPD(x) = \big(T(x), I(x), F(x)\big),$$

where:

- $T(x)$: The degree of truth (probability that x occurs).
- $I(x)$: The degree of indeterminacy (uncertainty whether x occurs or not).
- $F(x)$: The degree of falsity (probability that x does not occur).

These values satisfy the condition:

$$0 \leq T(x) + I(x) + F(x) \leq 3.$$

(2) *Neutrosophic Random Variable (NRV) [79, 174, 175]:* A random variable X in Neutrosophic Statistics maps outcomes to their corresponding neutrosophic probabilities:

$$X : \Omega \to \{(T, I, F) \in [0, 1]^3 \mid 0 \leq T + I + F \leq 3\}.$$

Unlike classical random variables, NRVs include indeterminate components, enabling more flexible representations of uncertainty.

(3) *Neutrosophic Expectation:* The expectation $E[X]$ of a neutrosophic random variable X is defined as:

$$E[X] = \big(E_T[X], E_I[X], E_F[X]\big),$$

where $E_T[X]$, $E_I[X]$, and $E_F[X]$ represent the expectations of truth, indeterminacy, and falsity, respectively.

(4) *Neutrosophic Variance:* The variance $\mathrm{Var}[X]$ is computed component-wise:

$$\mathrm{Var}[X] = \big(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]\big),$$

providing a detailed analysis of data uncertainty.

(5) *Neutrosophic Distribution Functions [2]:*

- The *Neutrosophic Probability Density Function (N-PDF)* is defined as:

$$f_X(x) = \big(f_T(x), f_I(x), f_F(x)\big),$$

where $f_T(x)$, $f_I(x)$, and $f_F(x)$ represent the densities for truth, indeterminacy, and falsity.

- The *Cumulative Neutrosophic Distribution Function (N-CDF)*:

$$F_X(x) = \big(F_T(x), F_I(x), F_F(x)\big),$$

accumulates probabilities across truth, indeterminacy, and falsity domains.

Example 3.13 (Neutrosophic Statistics: Disease Diagnosis)**.** Consider a medical diagnosis scenario (cf. [411]) where the sample space is $\Omega = \{\text{disease present}, \text{disease absent}\}$. Let $A = \{\text{disease present}\}$ represent the event of interest.

A Neutrosophic Probability Distribution (NPD) for A could be:

$$NPD(A) = (T(A), I(A), F(A)) = (0.7, 0.2, 0.1),$$

where:

- $T(A) = 0.7$: Probability that the patient truly has the disease.
- $I(A) = 0.2$: Uncertainty due to inconclusive test results.
- $F(A) = 0.1$: Probability that the patient does not have the disease.

Let a Neutrosophic Random Variable X represent the severity level of the disease on a scale from 0 to 10. For instance:

$$X(\text{disease present}) = (0.7, 0.2, 0.1), \quad X(\text{disease absent}) = (0.1, 0.3, 0.6).$$

Assuming severity level 8 for "disease present" and 0 for "disease absent," the Neutrosophic Expectation is:

$$E_T[X] = 0.7\times8+0.1\times0 = 5.6, \quad E_I[X] = 0.2\times8+0.3\times0 = 1.6, \quad E_F[X] = 0.1\times8+0.6\times0 = 0.8.$$

Thus, the expectation is:

$$E[X] = (5.6, 1.6, 0.8),$$

providing a comprehensive analysis of the severity with truth, indeterminacy, and falsity components.

Definition 3.14 (HyperNeutrosophic Statistics)**.** HyperNeutrosophic Statistics (HNS) is a framework that generalizes Neutrosophic Statistics by incorporating sets of Neutrosophic Probability Distributions (NPDs) to represent multiple sources of uncertainty or criteria simultaneously.

(1) *HyperNeutrosophic Probability Distribution (HNPD):* Let $x \in \Omega$ represent a random event. A HyperNeutrosophic Probability Distribution (HNPD) for x is defined as:

$$HNPD(x) = \{\big(T_i(x), I_i(x), F_i(x)\big) \mid i \in \mathcal{I}\},$$

where:

- $\mathcal{I}$ is an index set representing distinct sources, tests, or criteria.
- Each triple $\big(T_i(x), I_i(x), F_i(x)\big)$ satisfies:

$$0 \leq T_i(x), I_i(x), F_i(x) \leq 1 \quad \text{and} \quad 0 \leq T_i(x) + I_i(x) + F_i(x) \leq 3.$$

(2) *HyperNeutrosophic Random Variable (HNRV):* A HyperNeutrosophic Random Variable X maps outcomes to their corresponding HNPDs:

$$X : \Omega \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the power set of $[0,1]^3$, representing all possible sets of neutrosophic probabilities.

(3) *HyperNeutrosophic Expectation and Variance:* Let X be an HNRV. The expectation $E[X]$ and variance $\mathrm{Var}[X]$ are defined component-wise over all $i \in \mathcal{I}$:

$$E[X] = \big\{E_T[X], E_I[X], E_F[X]\big\}, \quad \text{where} \quad E_T[X] = \sum_{i \in \mathcal{I}} T_i(x),$$

and similarly for $E_I[X]$ and $E_F[X]$. The variance is given by:

$$\mathrm{Var}[X] = \big\{\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]\big\}.$$

Example 3.15 (HyperNeutrosophic Statistics: Multi-Test Disease Diagnosis)**.** Now consider the same scenario, but with three independent diagnostic tests $\mathcal{I} = \{1, 2, 3\}$.

A HyperNeutrosophic Probability Distribution (HNPD) for $A = \{\text{disease present}\}$ is:

$$HNPD(A) = \{(0.65, 0.25, 0.10), (0.75, 0.15, 0.10), (0.70, 0.20, 0.10)\},$$

where:

- Each triple $(T_i(A), I_i(A), F_i(A))$ corresponds to a test:
  - Test 1: $(0.65, 0.25, 0.10)$,
  - Test 2: $(0.75, 0.15, 0.10)$,
  - Test 3: $(0.70, 0.20, 0.10)$.

A HyperNeutrosophic Random Variable X maps outcomes to sets of triples. For "disease present," the set of probabilities is:

$$X(\text{disease present}) = HNPD(A).$$

Using severity level 8 for "disease present" and 0 for "disease absent," the aggregated expectations are:

$$E_T[X] = \frac{8 \times (0.65 + 0.75 + 0.70)}{3} = 5.6, \quad E_I[X] = \frac{8 \times (0.25 + 0.15 + 0.20)}{3} = 1.6, \quad E_F[X] = \frac{8 \times (0.10 + 0.10 + 0.10)}{3} = 0.8.$$

Thus, the HyperNeutrosophic Expectation is:

$$E[X] = (5.6, 1.6, 0.8),$$

capturing uncertainty across multiple diagnostic tests and providing a robust analysis.

Definition 3.16 (n-SuperHyperNeutrosophic Statistics)**.** n-SuperHyperNeutrosophic Statistics (n-SHNS) is a hierarchical generalization of HyperNeutrosophic Statistics (HNS). It introduces n-fold nested hyperstructures derived from Neutrosophic Probability Distributions (NPDs), allowing for multiple, recursively defined levels of uncertainty.

(1) *n-SuperHyperNeutrosophic Probability Distribution (n-SHNPD):* Let $x \in \Omega$ be a random event. A HyperNeutrosophic Probability Distribution (HNPD) is defined as a set of Neutrosophic Probability Distributions:

$$HNPD(x) = \{(T_i(x), I_i(x), F_i(x)) \mid i \in \mathcal{I}\},$$

where each triple $(T_i(x), I_i(x), F_i(x))$ corresponds to a neutrosophic characterization of x under the i-th source of uncertainty, with:

$$0 \leq T_i(x), I_i(x), F_i(x) \leq 1 \quad \text{and} \quad 0 \leq T_i(x) + I_i(x) + F_i(x) \leq 3.$$

An n-SuperHyperNeutrosophic Probability Distribution (n-SHNPD) is defined by applying the power set operation n times to the HNPD structure:

$$n\text{-SHNPD}(x) = \mathcal{P}^n\big(HNPD(x)\big),$$

where:

$$\mathcal{P}^1(HNPD(x)) = \mathcal{P}(HNPD(x)), \quad \mathcal{P}^2(HNPD(x)) = \mathcal{P}(\mathcal{P}(HNPD(x))), \quad \text{and so forth.}$$

Thus, for $n \geq 1$, n-SHNPD consists of n-fold nested sets of sets of neutrosophic triples, representing progressively more complex and layered uncertainty structures.

(2) *n-SuperHyperNeutrosophic Random Variable (n-SHNRV):* An n-SuperHyperNeutrosophic Random Variable X maps each outcome $\omega \in \Omega$ to an n-SHNPD:

$$X : \Omega \to n\text{-SHNPD}(x).$$

(3) *n-SuperHyperNeutrosophic Expectation and Variance:* The expectation $E[X]$ of an n-SHNRV is defined by recursively applying the expectation operator within the nested power sets, aggregating truth, indeterminacy, and falsity components at each level. Similarly, the variance $\mathrm{Var}[X]$ is defined by recursively computing and aggregating variances across all nested layers of uncertainty:

$$E[X] = \mathcal{P}^n\big(E_T[X], E_I[X], E_F[X]\big), \quad \mathrm{Var}[X] = \mathcal{P}^n\big(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]\big).$$

(4) *Recursive Structure:* For $k > 1$, a k-SuperHyperNeutrosophic Probability Distribution is defined as:

$$k\text{-SHNPD}(x) = \mathcal{P}((k-1)\text{-SHNPD}(x)).$$

By construction:

- For $n = 0$, n-SHNS reduces to Neutrosophic Statistics.
- For $n = 1$, it reduces to HyperNeutrosophic Statistics.
- For $n > 1$, it provides increasingly complex hierarchical structures to model uncertainty.

Theorem 3.17 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Statistics possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.18 (Reduction to Neutrosophic Statistics)**.** *For $n = 0$, n-SuperHyperNeutrosophic Statistics (SHNS) reduces to standard Neutrosophic Statistics.*

Proof. By definition, the n-SuperHyperNeutrosophic Probability Distribution (n-SHNPD) is constructed by applying n-fold nested power set operations to a HyperNeutrosophic Probability Distribution (HNPD). For $n = 0$, no power set operations are applied. This leaves us with the original Neutrosophic Probability Distribution (NPD), thus recovering the standard Neutrosophic Statistics framework as defined in [93, 360]. □

Theorem 3.19 (Strict Hierarchical Complexity)**.** *For every $n \geq 1$, the n-SHNPD is strictly more complex (in terms of hierarchical nesting) than the $(n-1)$-SHNPD.*

Proof. Consider $(n-1)$-SHNPD, defined as $(n-1)$-fold nested power sets of an HNPD. To obtain n-SHNPD, we apply one more power set operation:

$$n\text{-SHNPD}(x) = \mathcal{P}((n-1)\text{-SHNPD}(x)).$$

The power set operation $\mathcal{P}$ applied to any non-empty set S produces a strictly larger and more complex family of subsets. Thus, n-SHNPD is a strictly more nested and richer structure than

$(n-1)$-SHNPD. This increased complexity follows from the exponential growth in the number of subsets at each level of the hierarchy. □

Theorem 3.20 (Consistency of Expectation and Variance under Projection)**.** *Let X be an n-SuperHyperNeutrosophic Random Variable (n-SHNRV). The expectation $E[X]$ and variance Var$[X]$ defined at the n-th level are consistent with those at the $(n-1)$-th level under suitable projection.*

Proof. By definition, the expectation and variance at the n-th level, $E[X]$ and $\mathrm{Var}[X]$, are formed by applying the expectation and variance operations within nested power sets of neutrosophic triples:

$$E[X] = \mathcal{P}^n(E_T[X], E_I[X], E_F[X]), \quad \mathrm{Var}[X] = \mathcal{P}^n(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]).$$

If we consider a suitable "flattening" or projection operation π that reduces $\mathcal{P}^n$ to $\mathcal{P}^{n-1}$ by selecting representative elements or subsets at each level, the resulting structures satisfy:

$$\pi(E[X]) = \mathcal{P}^{n-1}(E_T[X], E_I[X], E_F[X]), \quad \pi(\mathrm{Var}[X]) = \mathcal{P}^{n-1}(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]).$$

These projected forms coincide with the expectations and variances defined at the $(n-1)$-th level. Hence, the operations are consistent across levels, maintaining hierarchical coherence of the statistical parameters. □

Theorem 3.21 (Non-emptiness of Expectation and Variance)**.** *For any n-SHNRV X, the sets $E[X]$ and Var$[X]$ are always well-defined and non-empty.*

Proof. Each level of the n-SHNRV is built upon non-empty sets of neutrosophic triples, as $HNPD(x)$ and its subsequent power sets $\mathcal{P}^k(HNPD(x))$ are by definition non-empty. The expectation and variance are computed by aggregating these neutrosophic values:

$$E[X] = \mathcal{P}^n(E_T[X], E_I[X], E_F[X]), \quad \mathrm{Var}[X] = \mathcal{P}^n(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X]).$$

Since $(E_T[X], E_I[X], E_F[X])$ and $(\mathrm{Var}_T[X], \mathrm{Var}_I[X], \mathrm{Var}_F[X])$ are derived from non-empty neutrosophic distributions, and the power set operation $\mathcal{P}$ applied to a non-empty set never results in an empty set, it follows that both $E[X]$ and $\mathrm{Var}[X]$ are well-defined and non-empty at every level n. □

Similarly, HyperFuzzy Statistics, HyperPlithogenic Statistics, SuperHyperFuzzy Statistics, and SuperHyperPlithogenic Statistics can also be defined.

Social Choice Theory is a branch of economics and political science that examines collective decision-making processes, where individual preferences are aggregated to determine a collective choice or societal welfare [39, 88, 269]. Decision-making, in this context, refers to the process of evaluating alternatives and selecting the most suitable option to achieve a desired objective [122, 123, 166, 259]. Extensions of traditional decision-making frameworks, such as Hyperdecision-Making and SuperHyperdecision-Making, have been developed to tackle complex systems involving hierarchical and multi-layered uncertainties [135]. Furthermore, decision-making has been extensively studied from perspectives such as Fuzzy and Neutrosophic frameworks, highlighting its importance across diverse domains [15, 16, 24, 25, 40, 270, 402, 445, 453].

In this paper, we extend Neutrosophic Decision Making to handle more intricate uncertainties by introducing HyperNeutrosophic Decision Making and SuperHyperNeutrosophic Decision Making, which are detailed in the following sections.

Definition 3.22 (Neutrosophic Decision Making (NDM))**.** [283] Neutrosophic Decision Making (NDM) is a multi-criteria decision-making framework that integrates neutrosophic logic, probability, and set theory to handle uncertainty, indeterminacy, and conflicting information.

Let $A = \{A_1, A_2, \dots, A_m\}$ be the set of m alternatives, and $C = \{C_1, C_2, \dots, C_n\}$ be the set of n criteria. Each alternative A_i is evaluated against each criterion C_j using a single-valued neutrosophic set (SVNS), represented as:

$$S_{ij} = \big(T_{ij}, I_{ij}, F_{ij}\big),$$

where:

- T_{ij} is the degree of truth that A_i satisfies C_j,
- I_{ij} is the degree of indeterminacy about A_i satisfying C_j,
- F_{ij} is the degree of falsity that A_i satisfies C_j.

These components satisfy the condition:

$$0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3, \quad \forall i, j.$$

Decision Matrix. The evaluations are organized into a decision matrix S:

$$S = \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1n} \\ S_{21} & S_{22} & \dots & S_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{m1} & S_{m2} & \dots & S_{mn} \end{bmatrix},$$

where each entry S_{ij} is a triple $\big(T_{ij}, I_{ij}, F_{ij}\big)$.

Weight Vector. Each criterion C_j is assigned a weight $w_j \in [0,1]$, representing its relative importance. The weights satisfy the normalization condition:

$$\sum_{j=1}^{n} w_j = 1.$$

Steps for Neutrosophic Decision Making. The steps for neutrosophic decision making are as follows:

1. Construct the decision matrix S using expert opinions or data, populating it with neutrosophic evaluations S_{ij}.
2. Identify the Ideal Neutrosophic Positive Solution (INPS) and Ideal Neutrosophic Negative Solution (INNS):
   - The INPS is given by:
   $$S^+ = \left(T_j^+, I_j^+, F_j^+\right), \quad T_j^+ = \max_i T_{ij},\ I_j^+ = \min_i I_{ij},\ F_j^+ = \min_i F_{ij}.$$
   - The INNS is given by:
   $$S^- = \left(T_j^-, I_j^-, F_j^-\right), \quad T_j^- = \min_i T_{ij},\ I_j^- = \max_i I_{ij},\ F_j^- = \max_i F_{ij}.$$
3. Compute the neutrosophic distance of each alternative A_i from both INPS and INNS using a distance metric such as Hamming or Euclidean distance.
4. Calculate the relative closeness of each alternative to INPS:
   $$RC_i = \frac{D_i^-}{D_i^+ + D_i^-},$$
   where D_i^+ and D_i^- are the distances of A_i from INPS and INNS, respectively.
5. Rank the alternatives based on their relative closeness RC_i. The alternative with the highest RC_i is the most preferred.

Definition 3.23 (HyperNeutrosophic Decision Making (HNDM))**.** HyperNeutrosophic Decision Making (HNDM) is an extension of Neutrosophic Decision Making (NDM) that incorporates a set of evaluations for each criterion, allowing for multiple perspectives or uncertain inputs to be represented simultaneously.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be the set of m alternatives, and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of n criteria. For each pair (A_i, C_j), the evaluation is represented as a set of triples:

$$H_{ij} = \left\{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \in \mathcal{K}_{ij}\right\},$$

where:

- $T_{ij}^k \in [0,1]$ is the degree of truth for the k-th evaluation,
- $I_{ij}^k \in [0,1]$ is the degree of indeterminacy for the k-th evaluation,
- $F_{ij}^k \in [0,1]$ is the degree of falsity for the k-th evaluation,

and $\mathcal{K}_{ij}$ is the index set of available evaluations for (A_i, C_j). Each triple satisfies:

$$0 \leq T_{ij}^k + I_{ij}^k + F_{ij}^k \leq 3, \quad \forall k \in \mathcal{K}_{ij}.$$

Steps for HNDM.

1. Construct the HyperNeutrosophic Decision Matrix H, where each entry H_{ij} is a set of evaluations.
2. Define the Ideal HyperNeutrosophic Positive Solution (IHNPS) and the Ideal Hyper-Neutrosophic Negative Solution (IHNNS) for each criterion C_j:

$$H_j^+ = \big\{ \max_k T_{ij}^k, \min_k I_{ij}^k, \min_k F_{ij}^k \big\},$$

$$H_j^- = \big\{ \min_k T_{ij}^k, \max_k I_{ij}^k, \max_k F_{ij}^k \big\}.$$

3. Compute the HyperNeutrosophic Distance for each alternative A_i from IHNPS and IHNNS.
4. Calculate the relative closeness to IHNPS for each alternative:

$$RC_i = \frac{D_i^-}{D_i^+ + D_i^-}.$$

5. Rank the alternatives based on RC_i.

Example 3.24 (Scenario: Medical Diagnosis and Treatment Plan Selection)**.** (cf. [167, 401, 433])

Problem: A patient requires a treatment plan for cancer (cf. [446]), and multiple diagnostic tests (cf. [114]) and expert opinions provide different perspectives on each option's effectiveness.

Setup:

- *Alternatives (A):* A_1: Chemotherapy [116], A_2: Radiation Therapy [57], A_3: Surgery [265].
- *Criteria (C):* C_1: Effectiveness, C_2: Side Effects, C_3: Cost.
- *Evaluations:* For each alternative A_i and criterion C_j, multiple evaluations are collected:

$$H_{ij} = \{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \text{ represents a test or expert opinion}\}.$$

  Example: For C_1 (Effectiveness of A_1), the results from three sources (e.g., imaging, biopsy [280], and expert review) could be:

$$H_{11} = \{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1), (0.85, 0.1, 0.05)\}.$$

Decision Process:

(1) Construct the HyperNeutrosophic Decision Matrix H with evaluations for each (A_i, C_j).

(2) Define the Ideal HyperNeutrosophic Positive Solution (IHNPS) to maximize effectiveness and minimize side effects and cost:

$$H_j^+ = \{ \max_k T_{ij}^k, \min_k I_{ij}^k, \min_k F_{ij}^k \}.$$

(3) Compute distances of each alternative A_i from IHNPS and Ideal HyperNeutrosophic Negative Solution (IHNNS).

(4) Rank the alternatives based on relative closeness to IHNPS.

Outcome: The treatment plan with the best balance of effectiveness, minimal side effects, and cost is chosen.

Definition 3.25 (n-SuperHyperNeutrosophic Decision Making (n-SHNDM))**.** n-SuperHyperNeutrosophic Decision Making (n-SHNDM) is a hierarchical generalization of HyperNeutrosophic Decision Making (HNDM), where the evaluations for each criterion are represented as nested sets of neutrosophic evaluations. This framework introduces n-fold nested hyperstructures derived from Neutrosophic Probability Distributions (NPDs), allowing for deeper levels of uncertainty, multi-layered inputs, and more complex modeling of preferences and criteria under uncertainty.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria. For each pair (A_i, C_j), the evaluation is represented as:

$$H_{ij}^{(n)} = \mathcal{P}^n\big([0,1]^3\big),$$

where:

- $n = 0$ reduces to Neutrosophic Decision Making, i.e., a single (T, I, F)-triple.
- $n = 1$ corresponds to HyperNeutrosophic Decision Making, i.e., a set of (T, I, F)-triples.
- $n > 1$ represents n-fold nested power sets of neutrosophic evaluations, capturing multiple layers of uncertainty and complexity.

Steps for n-SHNDM.

(1) Construct the n-SuperHyperNeutrosophic Decision Matrix $H^{(n)}$, where each entry $H_{ij}^{(n)}$ is an n-fold nested set of neutrosophic evaluations.

(2) Define the Ideal n-SuperHyperNeutrosophic Positive Solution (n-SHNPS) and the Ideal n-SuperHyperNeutrosophic Negative Solution (n-SHNNS):

$$H_j^+ = \max\big(H_{ij}^{(n)}\big), \quad H_j^- = \min\big(H_{ij}^{(n)}\big),$$

using suitable aggregation operators that account for the nested neutrosophic structure.

(3) Compute the n-SuperHyperNeutrosophic Distance for each alternative A_i from n-SHNPS and n-SHNNS.

(4) Calculate the relative closeness to n-SHNPS for each alternative:

$$RC_i = \frac{D_i^-}{D_i^+ + D_i^-}.$$

(5) Rank the alternatives based on RC_i.

Theorem 3.26 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Decision Making possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.27 (Reduction Property)**.** *For $n = 0$, n-SHNDM reduces to standard Neutrosophic Decision Making.*

Proof. When $n = 0$, no power set operations are applied. Hence, each evaluation $H_{ij}^{(0)}$ is simply a neutrosophic triple (T, I, F) directly representing the degree of truth, indeterminacy, and falsity for the criterion C_j applied to alternative A_i. This coincides with the standard Neutrosophic Decision Making model, proving the reduction property. □

Theorem 3.28 (Strict Hierarchical Complexity)**.** *For every $n \geq 1$, the evaluations in n-SHNDM are strictly more complex (hierarchically) than those in $(n-1)$-SHNDM.*

Proof. In $(n-1)$-SHNDM, each evaluation $H_{ij}^{(n-1)}$ is an $(n-1)$-fold nested power set of neutrosophic triples. To obtain n-SHNDM, we apply one additional power set operation:

$$H_{ij}^{(n)} = \mathcal{P}(H_{ij}^{(n-1)}).$$

The power set operation generates a strictly larger and more complex family of subsets. Thus, n-SHNDM inherently introduces a higher level of complexity than $(n-1)$-SHNDM. □

Theorem 3.29 (Existence of Ideal Solutions)**.** *For any n-SHNDM problem, the Ideal n-SuperHyperNeutrosophic Positive Solution (n-SHNPS) and the Ideal n-SuperHyperNeutrosophic Negative Solution (n-SHNNS) exist and are well-defined.*

Proof. Each $H_{ij}^{(n)}$ is a non-empty n-fold nested power set of neutrosophic triples. By the properties of power sets, these structures are always non-empty. Applying the max and min operations to these nested sets using appropriate neutrosophic aggregation operators yields non-empty results. Therefore, H_j^+ and H_j^- are always well-defined. Since this holds for all criteria C_j, the sets $\{H_j^+\}$ and $\{H_j^-\}$ form the n-SHNPS and n-SHNNS respectively, ensuring their existence. □

Theorem 3.30 (Continuity under Parameter Perturbations)**.** *If the neutrosophic evaluations (T, I, F) within the nested sets vary continuously with respect to parameter changes (e.g., weighting factors or external conditions), then the resulting rankings in n-SHNDM vary continuously as well.*

Proof. The decision-making process in n-SHNDM involves:

$$H_{ij}^{(n)} = \mathcal{P}^n([0,1]^3),$$

and subsequent aggregation steps (max, min, distance computations, and relative closeness) are compositions of continuous operations over compact sets in $[0,1]^3$. As continuous transformations of continuous parameterized inputs, the outputs (rankings) must also be continuous. Small perturbations in the input values yield small changes in the evaluated sets and, consequently, in the derived ranking, proving continuity. □

Theorem 3.31 (Monotonicity with Respect to Truth Degrees)**.** *Consider two alternatives A_p and A_q and a single criterion C_j. If every neutrosophic triple in $H_{pj}^{(n)}$ has greater or equal truth degrees T than the corresponding triples in $H_{qj}^{(n)}$ (with other factors equal), then A_p cannot rank lower than A_q.*

Proof. The decision-making framework ranks alternatives based on their relative closeness to the n-SHNPS and n-SHNNS, which ultimately depends on the values of T, I, and F within the nested sets. If $H_{pj}^{(n)}$ uniformly dominates $H_{qj}^{(n)}$ in truth (and all else is equal), the aggregated evaluation for A_p must be at least as favorable as that for A_q, preserving or improving A_p's relative closeness and thus ensuring A_p does not rank below A_q. □

Similarly, HyperFuzzy Decision Making, HyperPlithogenic Decision Making, SuperHyperFuzzy Decision Making, and SuperHyperPlithogenic Decision Making can also be defined.

3.4. *HyperNeutrosophic Language*

This subsection examines the concept of HyperNeutrosophic Language. Various linguistic frameworks, including Formal Languages [152, 190, 198, 315, 329] and Natural Languages [65, 98, 257], have been extensively explored in the literature. With advancements in fields such as machine learning, these studies have garnered significant attention, particularly in areas like Natural Language Processing (cf. [65, 98, 101, 168, 241, 257, 258, 301, 421, 432, 438, 444]). Technologies such as ChatGPT also incorporate these concepts in their applications ([7, 55, 133, 189, 212, 248]). Moreover, related ideas like HyperLanguage [69, 70, 132, 135, 144] and SuperHyperLanguage [135, 144] are well-documented and serve as foundational concepts for further exploration.

Related concepts, such as Fuzzy Language [108,109,305], Neutrosophic Language [144], and Plithogenic Language [144], have also been explored. In this context, we propose extending these frameworks to define HyperNeutrosophic Language and SuperHyperNeutrosophic Language. Their definitions and related concepts are outlined below.

Definition 3.32 (Neutrosophic Language)**.** [144] Let Σ be a finite alphabet. A *Neutrosophic Language* over Σ^* is a function:

$$N : \Sigma^* \to [0,1]^3,$$

where for each word $w \in \Sigma^*$, $N(w) = (T(w), I(w), F(w))$ with $T(w), I(w), F(w) \in [0,1]$ and

$$0 \leq T(w) + I(w) + F(w) \leq 3.$$

Here:

- $T(w)$ represents the *truth-membership degree* of w.
- $I(w)$ represents the *indeterminacy-membership degree* of w.
- $F(w)$ represents the *falsity-membership degree* of w.

A Neutrosophic Language generalizes the notion of membership beyond the single membership function of a fuzzy language by explicitly incorporating degrees of truth, indeterminacy, and falsity.

Example 3.33 (Applications of Neutrosophic Language in Real Life)**.** We present two practical examples of Neutrosophic Language.

1. Sentiment Analysis in Product Reviews: Consider a product review system [8,66] where $\Sigma = \{\text{excellent}, \text{good}, \text{average}, \text{poor}\}$. For each word w in the review, the Neutrosophic Language $N(w)$ assigns:

$$N(\text{excellent}) = (0.9, 0.05, 0.05), \quad N(\text{average}) = (0.5, 0.4, 0.1).$$

Here, $T(w)$ indicates the degree of positive sentiment, $I(w)$ represents uncertainty (e.g., mixed opinions), and $F(w)$ reflects negative sentiment. By incorporating these components, the system can perform nuanced analysis, considering both explicit sentiment and inherent ambiguities.

2. Medical Diagnosis in Healthcare: In a healthcare diagnostic system [348], let $\Sigma = \{\text{healthy}, \text{ill}, \text{uncertain}\}$. For a patient record word w, the Neutrosophic Language $N(w)$ assigns:

$$N(\text{healthy}) = (0.85, 0.1, 0.05), \quad N(\text{ill}) = (0.3, 0.5, 0.2).$$

Here, $T(w)$ quantifies the likelihood of being healthy, $I(w)$ captures uncertainty in the diagnosis, and $F(w)$ indicates the likelihood of being ill. This provides a structured framework for handling cases with incomplete or conflicting data.

Definition 3.34 (HyperNeutrosophic Language)**.** Let Σ be a finite alphabet. A *HyperNeutrosophic Language* over Σ^* is a function:

$$H : \Sigma^* \to \mathcal{P}([0,1]^3),$$

where for each word $w \in \Sigma^*$, $H(w) \subseteq [0,1]^3$ represents a set of neutrosophic triples:

$$H(w) = \{(T_k(w), I_k(w), F_k(w)) \mid k \in \mathcal{K}_w\},$$

where:

- $T_k(w), I_k(w), F_k(w) \in [0,1]$ are the k-th truth, indeterminacy, and falsity membership degrees, respectively.
- $\mathcal{K}_w$ is the index set of evaluations associated with w.
- Each triple satisfies:

$$0 \leq T_k(w) + I_k(w) + F_k(w) \leq 3.$$

The HyperNeutrosophic Language generalizes the Neutrosophic Language by allowing multiple evaluations for each word, enabling the representation of uncertain or conflicting linguistic data from different sources.

Example 3.35 (Applications of HyperNeutrosophic Language in Real Life)**.** This section presents two practical examples of HyperNeutrosophic Language.

1. Machine Translation with Multiple Evaluations: Consider a machine translation system [435] evaluated by multiple experts. Let $\Sigma = \{\text{hello}, \text{goodbye}\}$. For each word w, the HyperNeutrosophic Language $H(w)$ assigns:

$$H(\text{hello}) = \{(0.95, 0.03, 0.02), (0.90, 0.05, 0.05)\}, \quad H(\text{goodbye}) = \{(0.85, 0.1, 0.05), (0.8, 0.15, 0.05)\}.$$

Each triple corresponds to a different expert's evaluation, capturing variability in the perceived accuracy and reliability of translations.

2. Crowdsourced Content Moderation: In a content moderation system [406], let $\Sigma = \{\text{safe}, \text{offensive}\}$. For each word w, the HyperNeutrosophic Language $H(w)$ assigns:

$$H(\text{safe}) = \{(0.8, 0.1, 0.1), (0.85, 0.05, 0.1)\}, \quad H(\text{offensive}) = \{(0.2, 0.6, 0.2), (0.3, 0.5, 0.2)\}.$$

These evaluations reflect judgments from multiple moderators, allowing the system to incorporate diverse perspectives and handle ambiguous cases effectively.

Definition 3.36 (n-SuperHyperNeutrosophic Language)**.** Let Σ be a finite alphabet. An *n-SuperHyperNeutrosophic Language* over Σ^* is a function:

$$H^{(n)} : \Sigma^* \to \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ represents the n-th nested power set of $[0,1]^3$. For each word $w \in \Sigma^*$, the evaluation is recursively defined as:

$$H^{(n)}(w) = \begin{cases} [0,1]^3, & \text{if } n = 0, \\ \mathcal{P}(H^{(n-1)}(w)), & \text{if } n \geq 1. \end{cases}$$

In this framework:

- $n = 0$ corresponds to the basic Neutrosophic Language.
- $n = 1$ corresponds to the HyperNeutrosophic Language.
- $n > 1$ represents higher-order structures, where each level of nesting allows for increasingly complex representations of uncertainty and multi-layered evaluations.

Theorem 3.37 (Relation of an n-SuperHyperNeutrosophic Set)**.** *An n-SuperHyperNeutrosophic Language inherently exhibits the structure of an n-SuperHyperNeutrosophic Set.*

Proof. The statement follows directly and is self-evident. □

Theorem 3.38 (Generalization Property)**.** *For any $n \geq 0$, an n-SuperHyperNeutrosophic Language (n-SHNL) generalizes the concepts of Neutrosophic Language (NL) and HyperNeutrosophic Language (HNL). Specifically:*

(1) *For $n = 0$, n-SHNL reduces to a Neutrosophic Language.*

(2) *For $n = 1$, n-SHNL coincides with a HyperNeutrosophic Language.*

(3) *For $n > 1$, n-SHNL provides a hierarchical nesting of neutrosophic evaluations, extending beyond both NL and HNL.*

Proof. By Definition 3.36, we have:

$$H^{(n)} : \Sigma^* \to \mathcal{P}^n([0,1]^3).$$

Case $n = 0$: If $n = 0$, by definition:

$$H^{(0)}(w) = [0,1]^3,$$

which means for each word w, we assign a single triple $(T(w), I(w), F(w)) \in [0,1]^3$. This matches exactly the definition of a Neutrosophic Language (Definition 3.32), where each word w maps to a single neutrosophic triple. Thus, 0-SHNL = NL.

Case $n = 1$: If $n = 1$:

$$H^{(1)}(w) = \mathcal{P}(H^{(0)}(w)) = \mathcal{P}([0,1]^3).$$

This means for each word w, we assign a set of triples, each triple being a point in $[0,1]^3$. By Definition 3.34, this is exactly the definition of a HyperNeutrosophic Language. Hence, 1-SHNL = HNL.

Case $n > 1$: For $n > 1$,

$$H^{(n)}(w) = \mathcal{P}(H^{(n-1)}(w)).$$

Since $H^{(n-1)}(w)$ itself is formed by $(n-1)$-fold nesting of power sets of $[0,1]^3$, taking another power set increases the complexity and the hierarchy of the uncertainty representation. Thus, for $n > 1$, $H^{(n)}$ is a strict extension, allowing for n-layered uncertainty and complexity not captured by NL or HNL.

Combining these results, we see that n-SHNL generalizes both NL (at $n = 0$) and HNL (at $n = 1$), and provides greater complexity for $n > 1$. □

Theorem 3.39 (Reduction to Classical Languages)**.** *If for all words $w \in \Sigma^*$ and for all levels of nesting, the indeterminacy I and falsity F components vanish (i.e., $I(w) = 0$ and $F(w) = 0$ at the base level and consequently at all higher levels of nesting), then an n-SuperHyperNeutrosophic Language reduces to a classical crisp language membership function.*

Proof. Suppose for each $w \in \Sigma^*$ and at all levels of n-nesting, we have:

$$I(w) = 0, \quad F(w) = 0.$$

Under these conditions, every neutrosophic triple simplifies to:

$$(T(w), 0, 0),$$

where $T(w) \in [0, 1]$.

At $n = 0$ (the Neutrosophic Language level), $N(w) = (T(w), 0, 0)$ simply becomes a scalar membership in $[0, 1]$. If $T(w)$ is further restricted to be in $\{0, 1\}$, then it becomes a classical characteristic function defining a crisp language:

$$L = \{w \mid T(w) = 1\}.$$

For $n = 1$ (the HyperNeutrosophic level) or any $n > 1$, the nested sets of triples would also collapse to sets of $(T, 0, 0)$ values. Without indeterminacy or falsity, and assuming a crisp threshold, the entire nested structure does not add complexity. Ultimately, it boils down to a deterministic assignment, indistinguishable from a classical language membership if $T(w)$ is binary.

Thus, in the absence of indeterminacy and falsity, n-SHNL reduces to a classical language with crisp membership, confirming the claim. □

Theorem 3.40 (Monotonicity of Complexity)**.** *Let $H^{(n)}$ be an n-SuperHyperNeutrosophic Language. As n increases, the representational capacity and complexity of $H^{(n)}$ is non-decreasing. In particular, for all $m > n$, $H^{(m)}$ can represent at least as complex a structure as $H^{(n)}$.*

Proof. By definition:

$$H^{(n)}(w) \in \mathcal{P}^n([0,1]^3),$$

and

$$H^{(m)}(w) \in \mathcal{P}^m([0,1]^3).$$

Since $\mathcal{P}^m([0,1]^3)$ represents an m-fold power set of $[0,1]^3$, and $m > n$, we have:

$$\mathcal{P}^n([0,1]^3) \subseteq \mathcal{P}^m([0,1]^3).$$

This inclusion follows from the construction: each level of power set strictly expands or maintains the set of representable structures. No operation reduces complexity; at worst, it preserves it.

Thus, any neutrosophic evaluation representable at level n can be embedded into the representation at level $m > n$. Consequently, the complexity and representational capacity is non-decreasing as n grows, establishing monotonicity. □

Theorem 3.41 (Continuity of Representation). *If the base-level neutrosophic evaluations $(T(w), I(w), F(w))$ vary continuously in $[0,1]^3$ for each word w, then the resulting n-SuperHyperNeutrosophic Language $H^{(n)}$ changes continuously with respect to these variations at all nesting levels.*

Proof. At $n = 0$, the mapping $N(w) = (T(w), I(w), F(w))$ is trivially continuous as each component is a direct evaluation in $[0,1]$.

For $n = 1$, $H^{(1)}(w) = \mathcal{P}(N(w))$. The power set operator at this level represents sets of neutrosophic triples. Small continuous changes in $(T(w), I(w), F(w))$ at the base level reflect continuous changes in these sets (e.g., if sets are constructed by applying continuous selection criteria or aggregation functions).

By induction, suppose continuity holds for $H^{(n-1)}(w)$. The construction of $H^{(n)}(w) = \mathcal{P}(H^{(n-1)}(w))$ involves set operations and possibly aggregation operators that are continuous with respect to the underlying neutrosophic triples. Since composition of continuous operations remains continuous, $H^{(n)}(w)$ also varies continuously as $(T(w), I(w), F(w))$ at the base level change.

Hence, at every level of nesting, continuity is preserved. This proves that $H^{(n)}$ is a continuous extension of the base-level neutrosophic evaluations. □

Similarly, HyperFuzzy Language, HyperPlithogenic Language, SuperHyperFuzzy Language, and SuperHyperPlithogenic Language can also be defined.

Question 3.42. Can concepts such as SuperHyperPlithogenic HyperLanguage and SuperHyperPlithogenic SuperHyperLanguage, along with their related frameworks, be formally defined? What are their defining characteristics? Furthermore, is it possible to apply these concepts to natural language processing?

Question 3.43. Related concepts such as natural language generation [211, 238, 316], natural language understanding [36, 117, 303, 441], natural language inference [73, 94, 102, 232], and natural language interface [38, 194] have been extensively studied.

Is it possible to extend these ideas to define and explore concepts like Natural HyperLanguage, Natural SuperHyperLanguage, Natural HyperPlithogenic Language, and Natural SuperHyperPlithogenic Language? Could these extensions have practical applications in certain fields?

3.5. *HyperNeutrosophic Topological Space*

A Topological Space is a set of points with a topology, defining open sets to study continuity, convergence, and neighborhood relationships [30, 58, 89, 97, 204, 226]. Extended concepts such as Fuzzy Topological Space [91, 105, 160, 434], Neutrosophic Topological Space [31, 229], and Plithogenic Topological Space [9] are well-known. This paper further extends these to define HyperNeutrosophic Topological Space and SuperHyperNeutrosophic Topological Space.

Definition 3.44 (Neutrosophic Topological Space)**.** Let X be a non-empty set. A *Neutrosophic Topology (NT)* on X is a family τ of neutrosophic subsets of X satisfying the following axioms:

(1) $N_0, N_1 \in \tau$, where:

$$N_0 = \{(x, 0, 0, 0) \mid x \in X\}, \quad N_1 = \{(x, 1, 1, 1) \mid x \in X\}.$$

(2) The intersection of any two neutrosophic open sets is also neutrosophic open:

$$G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau.$$

(3) The union of any collection of neutrosophic open sets is neutrosophic open:

$$\{G_i \mid i \in J, G_i \in \tau\} \implies \bigcup_{i \in J} G_i \in \tau.$$

In this case, the pair (X, τ) is called a *Neutrosophic Topological Space (NTS)*, and any set in τ is referred to as a *neutrosophic open set (NOS)*(cf. [205, 312, 336]).

Definition 3.45 (HyperNeutrosophic Topological Space)**.** Let X be a non-empty set. A *HyperNeutrosophic Topology (HNT)* on X is a family τ_H of hyperneutrosophic subsets of X satisfying the following axioms:

(1) $N_0, N_1 \in \tau_H$, where:

$$N_0 = \{(x, \{(0,0,0)\}) \mid x \in X\}, \quad N_1 = \{(x, \{(1,1,1)\}) \mid x \in X\}.$$

(2) The intersection of any two hyperneutrosophic open sets is also hyperneutrosophic open:

$$G_1, G_2 \in \tau_H \implies G_1 \cap G_2 \in \tau_H.$$

(3) The union of any collection of hyperneutrosophic open sets is hyperneutrosophic open:

$$\{G_i \mid i \in J, G_i \in \tau_H\} \implies \bigcup_{i \in J} G_i \in \tau_H.$$

Each hyperneutrosophic set is defined as:

$$H(x) = \{(T_k(x), I_k(x), F_k(x)) \mid k \in \mathcal{K}_x\},$$

where:

- $T_k(x), I_k(x), F_k(x) \in [0,1]$ are the k-th truth, indeterminacy, and falsity degrees, respectively.
- $\mathcal{K}_x$ is the index set of evaluations for x.
- Each triple satisfies:

$$0 \leq T_k(x) + I_k(x) + F_k(x) \leq 3.$$

In this case, the pair (X, τ_H) is called a *HyperNeutrosophic Topological Space (H-NTS)*, and any set in τ_H is referred to as a *hyperneutrosophic open set*.

Definition 3.46 (n-SuperHyperNeutrosophic Topological Space)**.** Let X be a non-empty set. An *n-SuperHyperNeutrosophic Topology (n-S-HNT)* on X is a family $\tau_H^{(n)}$ of n-nested hyperneutrosophic subsets of X satisfying the following axioms:

(1) $N_0, N_1 \in \tau_H^{(n)}$, where:

$$N_0 = \{(x, \mathcal{P}^n(\{(0,0,0)\})) \mid x \in X\}, \quad N_1 = \{(x, \mathcal{P}^n(\{(1,1,1)\})) \mid x \in X\}.$$

(2) The intersection of any two n-superhyperneutrosophic open sets is also n-superhyperneutrosophic open:

$$G_1, G_2 \in \tau_H^{(n)} \implies G_1 \cap G_2 \in \tau_H^{(n)}.$$

(3) The union of any collection of n-superhyperneutrosophic open sets is n-superhyperneutrosophic open:

$$\{G_i \mid i \in J, G_i \in \tau_H^{(n)}\} \implies \bigcup_{i \in J} G_i \in \tau_H^{(n)}.$$

Each n-superhyperneutrosophic set is recursively defined as:

$$H^{(n)}(x) = \begin{cases} [0,1]^3, & \text{if } n = 0, \\ \mathcal{P}(H^{(n-1)}(x)), & \text{if } n \geq 1. \end{cases}$$

In this case, the pair $(X, \tau_H^{(n)})$ is called an *n-SuperHyperNeutrosophic Topological Space (n-S-HNTS)*, and any set in $\tau_H^{(n)}$ is referred to as an n-superhyperneutrosophic open set.

Theorem 3.47 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Topological Space possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.48 (Reduction to Neutrosophic Topological Spaces)**.** *If $n = 0$, then an n-SuperHyperNeutrosophic Topological Space (n-S-HNTS) reduces to a standard Neutrosophic Topological Space (NTS).*

Proof. For $n = 0$, we have:

$$H^{(0)}(x) = [0,1]^3,$$

representing a single neutrosophic triple $(T(x), I(x), F(x))$ with $0 \leq T(x) + I(x) + F(x) \leq 3$. No power set nesting is involved at this level, so the topology $\tau_H^{(0)}$ consists solely of neutrosophic subsets of X satisfying the usual neutrosophic topological axioms. Thus, $(X, \tau_H^{(0)})$ is precisely an NTS. □

Theorem 3.49 (Increasing Hierarchical Complexity)**.** *For every $n \geq 1$, an n-S-HNTS possesses strictly greater structural complexity than an $(n-1)$-S-HNTS.*

Proof. By construction, moving from level $(n-1)$ to n applies the power set operation $\mathcal{P}$:

$$H^{(n)}(x) = \mathcal{P}(H^{(n-1)}(x)).$$

The power set of a non-empty set is always strictly larger, introducing exponentially more subsets. Consequently, each step in this hierarchy adds a new layer of complexity, creating an increasingly intricate family of open sets. Hence, n-S-HNTS is more complex than $(n-1)$-S-HNTS in terms of the richness and combinatorial depth of its topological structure. □

Theorem 3.50 (Inductive Construction of n-S-HNTS)**.** *Suppose $(X, \tau_H^{(n-1)})$ is an $(n-1)$-SuperHyperNeutrosophic Topological Space. Define*

$$\tau_H^{(n)} = \{\mathcal{P}(G') \mid G' \in \tau_H^{(n-1)}\}.$$

Then $(X, \tau_H^{(n)})$ is an n-SuperHyperNeutrosophic Topological Space.

Proof. We use induction on n:

Base case $(n = 0)$*:* An 0-S-HNTS is just an NTS, which satisfies the neutrosophic topological axioms by definition.

Inductive step: Assume $(X, \tau_H^{(n-1)})$ is an $(n-1)$-S-HNTS. In particular:

(1) $N_0, N_1 \in \tau_H^{(n-1)}$.

(2) Finite intersections and arbitrary unions of sets in $\tau_H^{(n-1)}$ lie in $\tau_H^{(n-1)}$.

Define $\tau_H^{(n)} = \{\mathcal{P}(G') \mid G' \in \tau_H^{(n-1)}\}$.

- Since $N_0, N_1 \in \tau_H^{(n-1)}$, we have $\mathcal{P}(N_0), \mathcal{P}(N_1) \in \tau_H^{(n)}$. - For $G'_1, G'_2 \in \tau_H^{(n-1)}$, their intersection $G'_1 \cap G'_2$ lies in $\tau_H^{(n-1)}$. Thus:

$$\mathcal{P}(G'_1) \cap \mathcal{P}(G'_2) = \mathcal{P}(G'_1 \cap G'_2) \in \tau_H^{(n)}.$$

- For any collection $\{G'_i\}_{i \in J} \subseteq \tau_H^{(n-1)}$:

$$\bigcup_{i \in J} G'_i \in \tau_H^{(n-1)} \implies \bigcup_{i \in J} \mathcal{P}(G'_i) = \mathcal{P}\left(\bigcup_{i \in J} G'_i\right) \in \tau_H^{(n)}.$$

These verifications show that $\tau_H^{(n)}$ inherits the topological axioms from $\tau_H^{(n-1)}$. Therefore, $(X, \tau_H^{(n)})$ is indeed an n-S-HNTS. □

Definition 3.51 (Homomorphism)**.** (cf. [126,328]) A *homomorphism* is a structure-preserving map between two algebraic structures of the same type. For example, a function $f : A \to B$ is a homomorphism between two groups $(A, \cdot)$ and $(B, *)$ if for all $x, y \in A$:

$$f(x \cdot y) = f(x) * f(y).$$

Theorem 3.52 (Reduction Homomorphism)**.** *There exists a natural reduction mapping* $\rho_n : \tau_H^{(n)} \to \tau_H^{(n-1)}$ *that, given an* n*-S-HNTS, recovers the* $(n-1)$*-S-HNTS structure without violating the topological axioms.*

Proof. Define ρ_n to map each $G \in \tau_H^{(n)}$, where $G = \mathcal{P}(G')$ for some $G' \in \tau_H^{(n-1)}$, to the corresponding G'. This mapping is well-defined because every set at level n arises from taking a power set of a set at level $(n-1)$.

For any two sets $G_1, G_2 \in \tau_H^{(n)}$:

$$\rho_n(G_1 \cap G_2) = \rho_n(\mathcal{P}(G'_1) \cap \mathcal{P}(G'_2)) = \rho_n(\mathcal{P}(G'_1 \cap G'_2)) = G'_1 \cap G'_2 = \rho_n(G_1) \cap \rho_n(G_2).$$

For any collection $\{G_i\}_{i \in J} \subseteq \tau_H^{(n)}$:

$$\rho_n\left(\bigcup_{i \in J} G_i\right) = \rho_n\left(\mathcal{P}\left(\bigcup_{i \in J} G'_i\right)\right) = \bigcup_{i \in J} G'_i = \bigcup_{i \in J} \rho_n(G_i).$$

Thus, ρ_n respects both finite intersections and arbitrary unions, preserving the topological structure. This shows that ρ_n acts as a natural homomorphism, reducing an n-S-HNTS back to $(n-1)$-S-HNTS while maintaining all topological properties. □

Theorem 3.53 (Boundedness of the Hierarchy). *At each level n, the sets remain within $\mathcal{P}^n([0,1]^3)$, ensuring that the hierarchy of n-S-HNTS is bounded within the neutrosophic cube structure.*

Proof. Since each n-S-HNTS is constructed by repeatedly applying the power set operation to subsets of $[0,1]^3$, no element outside $[0,1]^3$ ever appears. The neutrosophic triples, and their nested power sets, all lie within a bounded domain. Hence, at any level n, the sets remain subsets of $\mathcal{P}^n([0,1]^3)$, ensuring that the entire hierarchy is contained within a finite, well-defined structure. □

Theorem 3.54 (Continuity under Nesting). *If the neutrosophic values (T, I, F) vary continuously within $[0,1]^3$, then the induced topologies at all levels n change continuously with respect to these values.*

Proof. The construction of $\tau_H^{(n)}$ from $\tau_H^{(n-1)}$ relies on the power set operation, which is purely set-theoretic and does not introduce discontinuities. Since the underlying neutrosophic triples are themselves defined within the continuous interval $[0,1]^3$, small perturbations in T, I, or F values affect the membership of sets in a continuous manner. Consequently, the resulting topologies at each level n vary continuously under small changes in the underlying neutrosophic values. □

Similarly, HyperFuzzy Topological Space, HyperPlithogenic Topological Space, SuperHyperFuzzy Topological Space, and SuperHyperPlithogenic Topological Space can also be defined.

3.6. *HyperNeutrosophic K-means Clustering*

K-means Clustering is an unsupervised machine learning algorithm that partitions data into k clusters by minimizing intra-cluster variance [191,203,209,222,342]. Extended concepts such as Fuzzy K-means Clustering [162,235,240,242,404,436] and Neutrosophic K-means Clustering [302, 319] are well-known. This paper further extends these ideas to HyperNeutrosophic K-means Clustering. The definition is provided below.

Definition 3.55 (Neutrosophic K-means Clustering)**.** [302, 319] Let $X = \{x_1, x_2, \ldots, x_n\}$ be a dataset, where each $x_i \in \mathbb{R}^d$. The goal of the Neutrosophic K-means algorithm is to partition X into k clusters $C_1, C_2, \ldots, C_k$ by minimizing the objective function:

$$J(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \mu_{C_j}(x_i)^m \|x_i - v_j\|^2,$$

where:

- $\mu_{C_j}(x_i)$ is the neutrosophic membership of x_i in cluster C_j, given as:

  $$\mu_{C_j}(x_i) = (T_{ij}, I_{ij}, F_{ij}),$$

  with $T_{ij}, I_{ij}, F_{ij} \in [0, 1]$ satisfying $0 \leq T_{ij} + I_{ij} + F_{ij} \leq 3$.
- $V = \{v_1, v_2, \ldots, v_k\}$ are the centroids of the clusters.
- $m > 1$ is a weighting exponent.

The algorithm iterates as follows:

1. Update the membership degrees:

$$\mu_{C_j}(x_i) = \left(\frac{\|x_i - v_j\|^{-2/(m-1)}}{\sum_{l=1}^{k} \|x_i - v_l\|^{-2/(m-1)}} \right).$$

2. Update the cluster centroids:

$$v_j = \frac{\sum_{x_i \in C_j} \mu_{C_j}(x_i)^m x_i}{\sum_{x_i \in C_j} \mu_{C_j}(x_i)^m}.$$

3. Repeat until convergence of $J(P, V)$.

Definition 3.56 (HyperNeutrosophic K-means Clustering)**.** The *HyperNeutrosophic K-means* extends the Neutrosophic K-means by allowing multiple neutrosophic membership evaluations for each data point. For each $x_i \in X$, the membership function is:

$$\mu_{C_j}(x_i) = \{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \in \mathcal{K}_{ij}\},$$

where $\mathcal{K}_{ij}$ is the index set of evaluations. Each $(T_{ij}^k, I_{ij}^k, F_{ij}^k)$ satisfies:

$$0 \leq T_{ij}^k + I_{ij}^k + F_{ij}^k \leq 3.$$

The objective function becomes:

$$J(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \left(\sum_{k \in \mathcal{K}_{ij}} \mu_{C_j}^k(x_i)^m \right) \|x_i - v_j\|^2.$$

The steps for updating $\mu_{C_j}(x_i)$ and v_j follow similar iterative updates as in the Neutrosophic K-means, applied over all evaluations k.

Definition 3.57 (n-SuperHyperNeutrosophic K-means Clustering)**.** The *n-SuperHyperNeutrosophic K-means* generalizes HyperNeutrosophic K-means by recursively nesting the membership evaluations. For each $x_i \in X$, the membership function is:

$$\mu_{C_j}^{(n)}(x_i) = \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ denotes the n-th power set of the neutrosophic unit cube. For $n = 0$, $\mu_{C_j}^{(0)}(x_i)$ reduces to a single neutrosophic triple. For $n = 1$, $\mu_{C_j}^{(1)}(x_i)$ corresponds to a set of triples as in HyperNeutrosophic K-means.

The objective function is recursively defined as:

$$J^{(n)}(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \left(\sum_{S \in \mu_{C_j}^{(n)}(x_i)} S^m \|x_i - v_j\|^2 \right).$$

The updates for $\mu_{C_j}^{(n)}(x_i)$ and v_j are similarly extended iteratively to handle the nested structure of the membership function.

Theorem 3.58 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic K-means Clustering possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.59 (Reduction to Neutrosophic K-means Clustering)**.** *If $n = 0$, then n-SuperHyperNeutrosophic K-means clustering coincides with standard Neutrosophic K-means clustering.*

Proof. For $n = 0$, the membership function reduces to:

$$\mu_{C_j}^{(0)}(x_i) = [0,1]^3,$$

which corresponds to a single neutrosophic triple (T_{ij}, I_{ij}, F_{ij}). Under these conditions, the objective function

$$J^{(0)}(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \mu_{C_j}(x_i)^m \|x_i - v_j\|^2$$

matches the standard Neutrosophic K-means objective. Consequently, centroid and membership updates follow the original Neutrosophic K-means algorithm. Thus, the 0-SuperHyperNeutrosophic K-means framework exactly replicates the standard Neutrosophic K-means approach. □

Theorem 3.60 (Hierarchical Complexity)**.** *For every $n \geq 1$, the n-SuperHyperNeutrosophic K-means framework is strictly more complex in membership representation than the $(n-1)$-SuperHyperNeutrosophic K-means framework.*

Proof. The membership function at level n is defined by:

$$\mu_{C_j}^{(n)}(x_i) = \mathcal{P}(\mu_{C_j}^{(n-1)}(x_i)),$$

where $\mathcal{P}$ denotes the power set. Applying the power set operation increases the number of subsets, thereby expanding the complexity of the membership structure. Since the power set of a non-empty set is strictly larger than the original set, the complexity of $\mu_{C_j}^{(n)}(x_i)$ strictly surpasses that of $\mu_{C_j}^{(n-1)}(x_i)$. This exponential growth in complexity propagates throughout the clustering process, making each successive level n more involved than the previous one. □

Theorem 3.61 (Continuity of the Objective Function)**.** *For any $n \geq 0$, the objective function $J^{(n)}(P, V)$ in n-SuperHyperNeutrosophic K-means clustering is continuous with respect to membership values and centroids.*

Proof. The objective function at level n is:

$$J^{(n)}(P, V) = \sum_{j=1}^{k} \sum_{x_i \in C_j} \left(\sum_{S \in \mu_{C_j}^{(n)}(x_i)} S^m \|x_i - v_j\|^2 \right).$$

Each term $S^m \|x_i - v_j\|^2$ involves elementary continuous operations (exponentiation, multiplication, and distance calculation). The membership values S and the centroids v_j enter these operations in a continuous manner. Since summations and power set aggregations preserve continuity, the entire objective function remains continuous. □

Theorem 3.62 (Convergence to a Local Minimum)**.** *The n-SuperHyperNeutrosophic K-means algorithm converges to a local minimum of the objective function $J^{(n)}(P, V)$.*

Proof. The algorithm updates membership values and centroids iteratively. Each update is designed to reduce the objective function or keep it constant, ensuring that the sequence of objective values is monotonically non-increasing and bounded below by zero. By the monotone convergence property, the algorithm must converge to a point where no further reduction is possible, i.e., a local minimum of $J^{(n)}(P, V)$. □

Theorem 3.63 (Partitioning Stability)**.** *For any fixed $n \geq 0$, the partitions obtained by n-SuperHyperNeutrosophic K-means clustering are stable under small perturbations of the input data.*

Proof. Small changes in the input data x_i produce correspondingly small changes in $\|x_i - v_j\|$. Since membership values depend continuously on these distances and are bounded, slight input variations lead to marginal adjustments in membership values and centroids. Thus, the cluster memberships and resulting partitions remain close to their original configuration, ensuring stability and robustness against minor data perturbations. □

Theorem 3.64 (Scalability of Complexity). *As n increases, the complexity of the n-SuperHyperNeutrosophic K-means algorithm grows exponentially with respect to the size of the membership sets.*

Proof. At level n, the membership structure involves $\mathcal{P}^n([0,1]^3)$, representing n-fold nested power sets. Each nesting layer multiplies the number of subsets, leading to an exponential increase in the cardinality of the membership representation. Consequently, computations involving membership aggregation, centroid updates, and objective function evaluations scale exponentially, reflecting a rapid growth in complexity as n increases. □

Theorem 3.65 (Generalization to Other Metrics). *The properties established for n-SuperHyperNeutrosophic K-means remain valid if the Euclidean norm $\|x_i - v_j\|$ is replaced by any continuous distance metric.*

Proof. All continuity, convergence, and stability arguments rely on the continuity and boundedness of the operations involved. Replacing the Euclidean norm with any other continuous distance metric preserves these properties. Since the definitions of membership updates, objective function computations, and centroid calculations depend only on continuity and boundedness, all theoretical guarantees extend naturally to other continuous metrics. □

Similarly, HyperFuzzy K-means, HyperPlithogenic K-means, SuperHyperFuzzy K-means, and SuperHyperPlithogenic K-means can also be defined. Additionally, extensions to related concepts such as C-means [6, 10, 183] can also be considered.

3.7. *Neutrosophic TOPSIS*

TOPSIS is a multi-criteria decision-making method selecting the best alternative by proximity to ideal and anti-ideal solutions [92, 127, 281]. As extensions of TOPSIS, methods like Fuzzy TOPSIS [181, 184, 186, 274, 282, 337, 405, 462] and Plithogenic TOPSIS [5, 260] are well-known. Here, we introduce Neutrosophic TOPSIS [4, 28, 218, 231, 244, 272, 273, 297]. Its definitions and related concepts are outlined below.

Definition 3.66 (Neutrosophic TOPSIS)**.** (cf. [244, 272, 273, 297]) Neutrosophic TOPSIS is a multi-criteria decision-making method utilizing neutrosophic triples (T, I, F) for each criterion. Let $A = \{A_1, \ldots, A_m\}$ be alternatives and $C = \{C_1, \ldots, C_n\}$ be criteria. The steps are as follows:

(1) Construct the decision matrix $D_{ij} = (T_{ij}, I_{ij}, F_{ij})$, where $T, I, F \in [0, 1]$ and $T+I+F \leq 3$.

(2) Normalize and weight the matrix. Normalize D_{ij} to obtain N_{ij}, then calculate the weighted normalized matrix as:

$$W_{ij} = w_j \cdot N_{ij},$$

where w_j is the weight of the j-th criterion.

(3) Define the Neutrosophic Positive Ideal Solution (NPIS) and Neutrosophic Negative Ideal Solution (NNIS):

$$NPIS = \left(T^+, I^-, F^-\right), \quad NNIS = \left(T^-, I^+, F^+\right),$$

where:

$$T^+ = \max_i W_{ij}(T), \quad I^- = \min_i W_{ij}(I), \quad F^- = \min_i W_{ij}(F),$$

$$T^- = \min_i W_{ij}(T), \quad I^+ = \max_i W_{ij}(I), \quad F^+ = \max_i W_{ij}(F).$$

(4) Calculate the separation measures from NPIS and NNIS:

$$S_i^+ = \sqrt{\sum_{j=1}^{n} \left((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\right)},$$

$$S_i^- = \sqrt{\sum_{j=1}^{n} \left((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\right)}.$$

(5) Compute the relative closeness of each alternative to the NPIS:

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}.$$

(6) Rank the alternatives based on the values of C_i, with higher C_i indicating better alternatives.

Definition 3.67 (HyperNeutrosophic TOPSIS)**.** Let $A = \{A_1, A_2, \ldots, A_m\}$ represent m alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ represent n criteria. In HyperNeutrosophic TOPSIS, each element of the decision matrix D is represented as a set of neutrosophic triples:

$$D_{ij} = \{(T_{ij}^k, I_{ij}^k, F_{ij}^k) \mid k \in \mathcal{K}_{ij}\},$$

where:

- $T_{ij}^k, I_{ij}^k, F_{ij}^k \in [0,1]$ are the k-th truth, indeterminacy, and falsity membership degrees for A_i under criterion C_j,
- $\mathcal{K}_{ij}$ is the index set of evaluations for D_{ij},
- Each triple satisfies:

$$0 \leq T_{ij}^k + I_{ij}^k + F_{ij}^k \leq 3.$$

The HyperNeutrosophic TOPSIS algorithm consists of the following steps:

(1) Construct the Decision Matrix. The decision matrix D consists of sets of neutrosophic triples for each criterion and alternative.

(2) Aggregate Membership Degrees. Aggregate the membership degrees over $k \in \mathcal{K}_{ij}$ using an appropriate aggregation operator $\oplus$:

$$\overline{D}_{ij} = \big(T_{ij}, I_{ij}, F_{ij}\big),$$

where:

$$T_{ij} = \oplus_{k\in\mathcal{K}_{ij}} T_{ij}^k, \quad I_{ij} = \oplus_{k\in\mathcal{K}_{ij}} I_{ij}^k, \quad F_{ij} = \oplus_{k\in\mathcal{K}_{ij}} F_{ij}^k.$$

(3) Normalize the Decision Matrix. Normalize $\overline{D}_{ij}$ using:

$$N_{ij} = \left(\frac{T_{ij}}{\max_i T_{ij}}, \frac{I_{ij}}{\max_i I_{ij}}, \frac{F_{ij}}{\max_i F_{ij}}\right).$$

(4) Weight the Normalized Decision Matrix. Apply weights w_j to obtain:

$$W_{ij} = w_j \cdot N_{ij}.$$

(5) Determine the Ideal Solutions. Compute the HyperNeutrosophic Positive Ideal Solution (HPIS) and HyperNeutrosophic Negative Ideal Solution (HNIS):

$$HPIS = \big(T^+, I^-, F^-\big), \quad HNIS = \big(T^-, I^+, F^+\big),$$

where:

$$T^+ = \max_i W_{ij}(T), \quad I^- = \min_i W_{ij}(I), \quad F^- = \min_i W_{ij}(F),$$
$$T^- = \min_i W_{ij}(T), \quad I^+ = \max_i W_{ij}(I), \quad F^+ = \max_i W_{ij}(F).$$

(6) Calculate Separation Measures. Compute the separation from HPIS and HNIS:

$$S_i^+ = \sqrt{\sum_{j=1}^n \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$
$$S_i^- = \sqrt{\sum_{j=1}^n \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

(7) Compute Relative Closeness. Calculate the relative closeness of each alternative to HPIS:

$$C_i = \frac{S_i^-}{S_i^+ + S_i^-}.$$

(8) Rank Alternatives. Rank alternatives based on C_i.

Definition 3.68 (n-SuperHyperNeutrosophic TOPSIS)**.** Let $A = \{A_1, A_2, \ldots, A_m\}$ and $C = \{C_1, C_2, \ldots, C_n\}$. In n-SuperHyperNeutrosophic TOPSIS, each element of the decision matrix D is recursively defined as:

$$D_{ij}^{(n)} = \mathcal{P}^n([0,1]^3),$$

where $\mathcal{P}^n([0,1]^3)$ represents the n-th power set of the neutrosophic unit cube.

The procedure extends HyperNeutrosophic TOPSIS with the following modifications:

(1) Recursive Aggregation. Aggregate the nested membership degrees recursively:

$$\overline{D}_{ij}^{(n)} = \text{Aggregate}\big(D_{ij}^{(n)}\big),$$

where the aggregation operator accounts for all nested layers.

(2) Normalize and Weight. Normalize and weight $\overline{D}_{ij}^{(n)}$ as in HyperNeutrosophic TOPSIS.

(3) Ideal Solutions. Define n-SuperHyperNeutrosophic Positive Ideal Solution (n-SHPIS) and Negative Ideal Solution (n-SHNIS) recursively.

(4) Separation Measures. Extend the separation measures to nested structures:

$$S_i^{+,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$

$$S_i^{-,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

(5) Relative Closeness. Compute the relative closeness for nested structures:

$$C_i^{(n)} = \frac{S_i^{-,(n)}}{S_i^{+,(n)} + S_i^{-,(n)}}.$$

(6) Rank Alternatives. Rank based on $C_i^{(n)}$.

Theorem 3.69 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic TOPSIS possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.70 (Reduction to Neutrosophic TOPSIS)**.** *If $n = 0$, then n-SuperHyperNeutrosophic TOPSIS reduces to standard Neutrosophic TOPSIS.*

Proof. When $n = 0$, each element $D_{ij}^{(0)}$ of the decision matrix is simply a neutrosophic triple (T_{ij}, I_{ij}, F_{ij}) without any additional nesting. In this case, all steps of the method—normalization, weighting, computation of NPIS/NNIS, separation measures, relative closeness, and ranking—are identical to those of the standard Neutrosophic TOPSIS procedure.

Thus, the 0-SuperHyperNeutrosophic TOPSIS framework coincides exactly with Neutrosophic TOPSIS. □

Theorem 3.71 (Hierarchical Complexity)**.** *For every $n \geq 1$, the n-SuperHyperNeutrosophic TOPSIS framework is strictly more complex in terms of aggregation and computation than the $(n-1)$-SuperHyperNeutrosophic TOPSIS framework.*

Proof. The n-SuperHyperNeutrosophic decision matrix $D_{ij}^{(n)}$ is defined recursively by:

$$D_{ij}^{(n)} = \mathcal{P}(D_{ij}^{(n-1)}),$$

where $\mathcal{P}$ denotes the power set operation. Each application of $\mathcal{P}$ produces an exponentially larger set of subsets, thereby increasing both the complexity of the membership structures and the effort required to aggregate them. Hence, as n increases, the amount of computational work grows significantly, making n-SuperHyperNeutrosophic TOPSIS strictly more complex than its $(n-1)$-level counterpart. □

Theorem 3.72 (Continuity of Relative Closeness)**.** *The relative closeness $C_i^{(n)}$ in n-SuperHyperNeutrosophic TOPSIS is a continuous function of the membership values and the criterion weights.*

Proof. The relative closeness for the n-level framework is given by:

$$C_i^{(n)} = \frac{S_i^{-,(n)}}{S_i^{+,(n)} + S_i^{-,(n)}},$$

where

$$S_i^{+,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^+)^2 + (I_{ij} - I^-)^2 + (F_{ij} - F^-)^2\big)},$$

and

$$S_i^{-,(n)} = \sqrt{\sum_{j=1}^{n} \big((T_{ij} - T^-)^2 + (I_{ij} - I^+)^2 + (F_{ij} - F^+)^2\big)}.$$

Each component (T_{ij}, I_{ij}, F_{ij}) is drawn from a continuous domain $[0,1]^3$. Arithmetic operations, squaring, and taking square roots are all continuous mappings. Therefore, $S_i^{+,(n)}$ and $S_i^{-,(n)}$ depend continuously on the membership values and weights. As a ratio of two continuous, strictly positive expressions, $C_i^{(n)}$ itself is continuous. □

Theorem 3.73 (Convergence)**.** *The n-SuperHyperNeutrosophic TOPSIS method converges to a stable ranking of alternatives as the iterative process reaches a steady state.*

Proof. The iterative steps of the algorithm involve standardizable processes: normalization, weighting, and aggregation of nested memberships. Since each neutrosophic value (T, I, F) is bounded within $[0, 1]$, these transformations remain stable and well-defined. As iteration continues, successive recalculations of separation measures and relative closeness values approach fixed points, leading the algorithm to stabilize. Consequently, the final ranking of alternatives will not oscillate indefinitely but will converge to a stable ordering. □

Theorem 3.74 (Consistency of Ideal Solutions). *The n-SuperHyperNeutrosophic Positive Ideal Solution (n-SHPIS) and Negative Ideal Solution (n-SHNIS) consistently reflect the extreme preferences defined by the weighted, normalized decision matrix* $D_{ij}^{(n)}$.

Proof. The n-SHPIS and n-SHNIS (T^+, I^-, F^-) and (T^-, I^+, F^+) are determined by extremizing the criterion values in the weighted normalized space. Because these extremizing operations directly use the aggregated values at the n-th level, they yield consistent definitions of "best" and "worst" scenarios with respect to truth, indeterminacy, and falsity dimensions. Thus, the ideal solutions accurately represent the extremal preference structure within the n-SuperHyperNeutrosophic framework, ensuring coherent and meaningful comparisons between alternatives. □

Similarly, HyperFuzzy TOPSIS, HyperPlithogenic TOPSIS, SuperHyperFuzzy TOPSIS, and SuperHyperPlithogenic TOPSIS can also be defined.

3.8. *Neutrosophic Evolution*

Darwin's Evolution explains the process of adaptation and evolution driven by natural selection [59, 125, 176, 177, 320, 430]. In contrast, Neutrosophic Evolution provides a multidimensional framework that incorporates truth, falsity, and indeterminacy to model adaptation and change [99, 286, 287, 372, 376]. This concept can also be interpreted using HyperNeutrosophic Sets or n-SuperHyperNeutrosophic Sets. The following sections outline their definitions and related formulations.

Definition 3.75 (Neutrosophic Evolution). [376] Neutrosophic Evolution describes the adaptation of a being B (e.g., plant, animal, human, or system) in a given environment η over a specific timespan Δt. It considers three components for every part P of B:

$$E(P), I(P), \text{ and } N(P),$$

representing degrees of evolution, involution, and neutrality (indeterminacy), respectively.

Mathematical Representation. For a being B with parts $P_1, P_2, \ldots, P_n$, the state of each part P_i is defined as a Neutrosophic Triple:

$$S(P_i) = \big(E(P_i), I(P_i), N(P_i)\big),$$

where:

- $E(P_i) \in [0,1]$: Degree of evolution of part P_i,
- $I(P_i) \in [0,1]$: Degree of involution (degeneration) of part P_i,
- $N(P_i) \in [0,1]$: Degree of neutrality (indeterminate change) of part P_i,

and the following constraint holds:

$$0 \leq E(P_i) + I(P_i) + N(P_i) \leq 3.$$

Total Neutrosophic State of a Being. The overall state of B at time t is represented as:

$$S(B,t) = \Big(E(B,t), I(B,t), N(B,t)\Big),$$

where:

$$E(B,t) = \frac{1}{n}\sum_{i=1}^{n} E(P_i), \quad I(B,t) = \frac{1}{n}\sum_{i=1}^{n} I(P_i), \quad N(B,t) = \frac{1}{n}\sum_{i=1}^{n} N(P_i).$$

Dynamic Behavior. The evolution of B over time t is defined by the derivatives:

$$\frac{\partial E(B,t)}{\partial t}, \quad \frac{\partial I(B,t)}{\partial t}, \quad \frac{\partial N(B,t)}{\partial t}.$$

These represent the rates of change of evolution, involution, and neutrality, respectively. The dynamic system governing $S(B,t)$ is expressed as:

$$\frac{\partial E(B,t)}{\partial t} = f_1(E, I, N, \eta, \Delta t),$$
$$\frac{\partial I(B,t)}{\partial t} = f_2(E, I, N, \eta, \Delta t),$$
$$\frac{\partial N(B,t)}{\partial t} = f_3(E, I, N, \eta, \Delta t),$$

where f_1, f_2, f_3 are functions modeling the influence of environmental factors η and time Δt.
Applications. This framework can be applied to:

(1) Biological adaptation (e.g., species evolution and regression).
(2) Societal systems (e.g., cultural growth and decline).
(3) Artificial systems (e.g., machine learning model optimization and decay).

Example 3.76 (Neutrosophic Evolution in Real Life)**.** Consider the evolution of a city's traffic system B in response to the introduction of electric vehicles (EVs [129,250]) in an environment η with fluctuating adoption rates and policies over a period Δt. Each component P_i of the traffic system (e.g., road infrastructure, charging stations, traffic laws) is characterized by:

$$S(P_i) = \big(E(P_i), I(P_i), N(P_i)\big),$$

where:

- $E(P_i)$: The degree to which the component evolves (e.g., expansion of charging stations).
- $I(P_i)$: The degree of indeterminacy (e.g., uncertainty in policy implementation).
- $N(P_i)$: The degree of neutrality (e.g., components unaffected by EV adoption, like pedestrian crossings [233]).

For instance:

$$S(\text{Charging Stations}) = (0.9, 0.1, 0.0), \quad S(\text{Road Markings}) = (0.4, 0.3, 0.3), \quad S(\text{Traffic Laws}) = (0.7, 0.2, 0.1).$$

The total Neutrosophic state of the traffic system is:

$$S(B,t) = \Big(E(B,t), I(B,t), N(B,t)\Big),$$

where:

$$E(B,t) = \frac{1}{3}(0.9 + 0.4 + 0.7) = 0.67, \quad I(B,t) = \frac{1}{3}(0.1 + 0.3 + 0.2) = 0.2, \quad N(B,t) = 0.13.$$

This analysis helps city planners understand the degree of adaptation, uncertainty, and neutral impacts in the system.

Neutrosophic Evolution can be applied to a wide range of concepts. Several examples are provided below.

Example 3.77 (Examples of Neutrosophic Evolution)**.** The concept of Neutrosophic Evolution can be intuitively understood through the following real-world examples:

(1) *Biological Evolution in Urban Birds (cf. [349]):* In cities, bird species such as pigeons exhibit behavioral and physical adaptations (evolution) to urban environments, like foraging near human activity or navigating buildings. However, some traditional behaviors (e.g., nesting in natural habitats) might diminish (involution), while certain traits, such as tolerance to human presence, remain neutral (indeterminate).

(2) *Cultural Dynamics in a Society:* A society's language (cf. [163]) evolves by incorporating new slang or technical terms (evolution), while older, less-used words or traditions fade (involution). Neutral words or practices that are neither widely adopted nor completely abandoned, like regional dialects, remain in an indeterminate state.

(3) *Adaptation of Machine Learning Models:* A machine learning model trained for spam detection updates its parameters when new patterns of spam emails emerge [182] (evolution). Outdated features lose their relevance and are discarded (involution). Features that are rarely triggered by current data may stay in a neutral state until more evidence supports their inclusion or exclusion (indeterminate).

(4) *Ecosystem Response to Climate Change:* Coral reefs adapt to rising temperatures by developing heat-tolerant algae (evolution). Some species, unable to cope, face population decline or extinction (involution). Other species might exhibit no clear trend of adaptation or decline due to insufficient data or complexity of interactions (indeterminacy).

(5) *Technological Development in Renewable Energy:* Solar panel technology [417] improves with higher efficiency rates and cost reductions (evolution). Outdated manufacturing methods become obsolete (involution). Experimental methods, such as new types of photovoltaic cells, remain under investigation, representing an indeterminate state until proven effective or ineffective.

Definition 3.78 (HyperNeutrosophic Evolution)**.** HyperNeutrosophic Evolution extends Neutrosophic Evolution by incorporating multiple evaluations for each component (evolution, involution, and neutrality) of a being B. For each part P of B, the state is represented as:

$$H(P) = \{(E_k(P), I_k(P), N_k(P)) \mid k \in \mathcal{K}_P\},$$

where:

- $E_k(P), I_k(P), N_k(P) \in [0,1]$ are the k-th evaluations of evolution, involution, and neutrality for P,
- $\mathcal{K}_P$ is the index set of evaluations for P,
- Each triple satisfies:

$$0 \leq E_k(P) + I_k(P) + N_k(P) \leq 3.$$

Total HyperNeutrosophic State of a Being. The overall state of B at time t is given by:

$$H(B,t) = \Big(H_E(B,t), H_I(B,t), H_N(B,t)\Big),$$

where:

$$H_E(B,t) = \frac{1}{n}\sum_{i=1}^{n} \Big\{ \oplus_{k\in\mathcal{K}_{P_i}} E_k(P_i)\Big\}, \quad H_I(B,t) = \frac{1}{n}\sum_{i=1}^{n} \Big\{ \oplus_{k\in\mathcal{K}_{P_i}} I_k(P_i)\Big\},$$

$$H_N(B,t) = \frac{1}{n}\sum_{i=1}^{n} \Big\{ \oplus_{k\in\mathcal{K}_{P_i}} N_k(P_i)\Big\}.$$

Here, $\oplus$ is an aggregation operator over multiple evaluations.

Dynamic Behavior. The rates of change are defined as:

$$\frac{\partial H_E(B,t)}{\partial t}, \quad \frac{\partial H_I(B,t)}{\partial t}, \quad \frac{\partial H_N(B,t)}{\partial t}.$$

These are governed by:

$$\begin{aligned}
\frac{\partial H_E(B,t)}{\partial t} &= f_1(H_E, H_I, H_N, \eta, \Delta t), \\
\frac{\partial H_I(B,t)}{\partial t} &= f_2(H_E, H_I, H_N, \eta, \Delta t), \\
\frac{\partial H_N(B,t)}{\partial t} &= f_3(H_E, H_I, H_N, \eta, \Delta t).
\end{aligned}$$

Example 3.79 (HyperNeutrosophic Evolution in Real Life)**.** Consider the adaptation of an international healthcare system B during a global pandemic (cf. [266,455]). Each component P_i of the system (e.g., hospital infrastructure, vaccination programs [77], public health campaigns [431]) is evaluated across multiple regions or agencies $k \in \mathcal{K}_{P_i}$, leading to:

$$H(P_i) = \{(E_k(P_i), I_k(P_i), N_k(P_i)) \mid k \in \mathcal{K}_{P_i}\}.$$

For example:

$$H(\text{Hospitals}) = \{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1)\}, \quad H(\text{Vaccination Program}) = \{(0.9, 0.05, 0.05), (0.85, 0.1, 0.05)\}.$$

The total HyperNeutrosophic state of the healthcare system is:

$$H(B,t) = \Big(H_E(B,t), H_I(B,t), H_N(B,t)\Big),$$

where:

$$H_E(B,t) = \frac{1}{2}\{(0.8 \oplus 0.7), (0.9 \oplus 0.85)\} = \{0.75, 0.875\},$$

and similarly for $H_I(B,t)$ and $H_N(B,t)$. Here, $\oplus$ represents an aggregation operator, such as averaging or maximum selection.

This analysis allows stakeholders to understand how different components of the healthcare system adapt under varying regional constraints, uncertainties, and neutral factors, providing a comprehensive framework for policy-making and resource allocation.

Definition 3.80 (n-SuperHyperNeutrosophic Evolution)**.** n-SuperHyperNeutrosophic Evolution is a hierarchical generalization of HyperNeutrosophic Evolution. For each part P of B, the state is recursively defined as:

$$H^{(n)}(P) = \mathcal{P}^n([0,1]^3),$$

where:

- $\mathcal{P}^n([0,1]^3)$ is the n-th nested power set of $[0,1]^3$,
- $H^{(0)}(P) = (E(P), I(P), N(P))$ as in Neutrosophic Evolution,
- $H^{(1)}(P) = H(P)$ as in HyperNeutrosophic Evolution.

Total n-SuperHyperNeutrosophic State of a Being. The total state of B is:

$$H^{(n)}(B,t) = \Big(H_E^{(n)}(B,t), H_I^{(n)}(B,t), H_N^{(n)}(B,t)\Big),$$

where:

$$H_E^{(n)}(B,t) = \frac{1}{n}\sum_{i=1}^{n} \text{Aggregate}\big(H_E^{(n-1)}(P_i)\big),$$

and similar definitions for $H_I^{(n)}(B,t)$ and $H_N^{(n)}(B,t)$.

Dynamic Behavior. The dynamic system for n-SuperHyperNeutrosophic Evolution is:

$$\frac{\partial H_E^{(n)}(B,t)}{\partial t} = f_1^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_I^{(n)}(B,t)}{\partial t} = f_2^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_N^{(n)}(B,t)}{\partial t} = f_3^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t).$$

Theorem 3.81 (Relation of an n-SuperHyperNeutrosophic Set)**.** *n-SuperHyperNeutrosophic Evolution possesses the structure of an n-SuperHyperNeutrosophic Set.*

Proof. This follows directly and is evident. □

Theorem 3.82 (Reduction to Neutrosophic Evolution)**.** *For $n = 0$, n-SuperHyperNeutrosophic Evolution reduces to standard Neutrosophic Evolution.*

Proof. By definition, 0-SuperHyperNeutrosophic Evolution means no power set operation is applied. Thus, for each part P of the being B:

$$H^{(0)}(P) = (E(P), I(P), N(P)),$$

which corresponds exactly to the Neutrosophic Evolution definition. The average measures $E(B,t), I(B,t), N(B,t)$ and their dynamics are then identical to those in the original Neutrosophic Evolution framework. Hence, 0-SHN Evolution coincides with Neutrosophic Evolution.

□

Theorem 3.83 (Strict Hierarchical Complexity)**.** *For every $n \geq 1$, n-SuperHyperNeutrosophic Evolution is strictly more complex than $(n-1)$-SuperHyperNeutrosophic Evolution in terms of state representation.*

Proof. The state representation for each part P at the n-th level involves:

$$H^{(n)}(P) = \mathcal{P}(H^{(n-1)}(P)),$$

where $\mathcal{P}$ is the power set operation. Since $\mathcal{P}$ applied to a non-empty set yields a strictly larger and more complex family of subsets, the complexity of the state representation grows as n increases. Each level n introduces an exponentially richer structure of nested sets of triples (E, I, N). Hence, the complexity of $H^{(n)}(P)$ strictly exceeds that of $H^{(n-1)}(P)$. □

Theorem 3.84 (Existence and Uniqueness of Solutions)**.** *Assume the functions $f_1^{(n)}, f_2^{(n)}, f_3^{(n)}$ governing the n-SuperHyperNeutrosophic Evolution dynamics are continuous and satisfy Lipschitz conditions in the variables $(H_E^{(n)}, H_I^{(n)}, H_N^{(n)})$. Then, for given initial conditions $(H_E^{(n)}(B,0), H_I^{(n)}(B,0), H_N^{(n)}(B,0))$, there exists a unique local solution for the dynamical system:*

$$\frac{\partial H_E^{(n)}(B,t)}{\partial t} = f_1^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_I^{(n)}(B,t)}{\partial t} = f_2^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t),$$
$$\frac{\partial H_N^{(n)}(B,t)}{\partial t} = f_3^{(n)}(H_E^{(n)}, H_I^{(n)}, H_N^{(n)}, \eta, \Delta t).$$

Proof. The system of partial differential equations in time can be reduced to an equivalent system of ordinary differential equations if we consider spatial variables fixed or if the model is spatially lumped. Under standard ODE existence and uniqueness theorems (such as Picard–Lindelöf), continuity and Lipschitz conditions on $f_1^{(n)}, f_2^{(n)}, f_3^{(n)}$ ensure a unique local solution exists through the given initial state. Thus, standard ODE theory applies, guaranteeing existence and uniqueness. □

Theorem 3.85 (Continuity under Parameter Perturbations)**.** *If the functions $f_1^{(n)}, f_2^{(n)}, f_3^{(n)}$ and the aggregation operators used at level n are continuous, then the solution $(H_E^{(n)}(B,t), H_I^{(n)}(B,t), H_N^{(n)}(B,t))$ depends continuously on environmental parameters η and time scale Δt.*

Proof. Since the state $(H_E^{(n)}, H_I^{(n)}, H_N^{(n)})$ at each time t is obtained by integrating a continuous system of differential equations with continuous initial conditions and parameters, small perturbations in η or Δt yield correspondingly small changes in the solutions. Continuity of the aggregation operators ensures that nested evaluations do not introduce discontinuities. Therefore, the entire n-SHN Evolution framework is stable under small parameter perturbations.

□

Theorem 3.86 (Boundedness of the State Space)**.** *For each part P and each triple $(E, I, N) \in [0,1]^3$, the nested structures remain within $\mathcal{P}^n([0,1]^3)$. Therefore, the state space for n-SHN Evolution is bounded.*

Proof. By construction, each neutrosophic triple (E, I, N) satisfies $E, I, N \in [0, 1]$. The power set operation preserves boundedness as it does not introduce elements outside the original set. Hence, for all n, the state $H^{(n)}(P) \subseteq \mathcal{P}^n([0, 1]^3)$ is confined to a bounded domain. Consequently, $H^{(n)}(B, t)$ remains within a bounded state space for all t. □

Similarly, HyperFuzzy Evolution, HyperPlithogenic Evolution, SuperHyperFuzzy Evolution, and SuperHyperPlithogenic Evolution can also be defined.

Funding

No external funding was received for this research.

Acknowledgments

We extend our heartfelt thanks to everyone who provided invaluable support in completing this paper; this work would not have been possible without the contributions of all those involved. We also express our deep respect and gratitude to the authors of prior studies cited in our references. Lastly, we are sincerely thankful to our readers for their time and interest.

Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[2] Hareem Abbasi, Usman Shahzad, Walid Emam, Muhammad Hanif, Nasir Ali, and Mubeen Mukhtar. Calibrated empirical neutrosophic cumulative distribution function estimation for both symmetric and asymmetric data. *Symmetry*, 16:633, 2024.

[3] Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.

[4] Mohamed Abdel-Basset, Mai Mohamed, and Florentin Smarandache. A hybrid neutrosophic group anp-topsis framework for supplier selection problems. *Symmetry*, 10:226, 2018.

[5] Mohamed Abdel-Basset and Rehab Mohamed. A novel plithogenic topsis-critic model for sustainable supply chain risk management. *Journal of Cleaner production*, 247:119586, 2020.

[6] Ahmed Abdelhafeez, Hoda K Mohamed, Ali Maher, and Ahmed Abdelmonem. A neutrosophic based c-means approach for improving breast cancer clustering performance. *Neutrosophic Sets and Systems*, 53:317–330, 2023.

[7] Josh Achiam, Steven Adler, Sandhini Agarwal, Lama Ahmad, Ilge Akkaya, Florencia Leoni Aleman, Diogo Almeida, Janko Altenschmidt, Sam Altman, Shyamal Anadkat, et al. Gpt-4 technical report. *arXiv preprint arXiv:2303.08774*, 2023.

[8] J K Adarsh, V. T. Sreedevi, and Deepa Thangavelusamy. Product review system with bert for sentiment analysis and implementation of administrative privileges on node-red. *IEEE Access*, 11:65968–65976, 2023.

[9] Nehmat Ahmed and Osama T. Pirbal. Plithogenic crisp hypersoft topology. *European Journal of Pure and Applied Mathematics*, 2024.

[10] Yaman Akbulut, Abdulkadir Şengür, Yanhui Guo, and Kemal Polat. Kncm: Kernel neutrosophic c-means clustering. *Applied Soft Computing*, 52:714–724, 2017.

[11] Charles O Akinyokun, Okure U Obot, Faith-Michael E Uzoka, and John J Andy. A neuro-fuzzy decision support system for the diagnosis of heart failure. In *Medical and care compunetics 6*, pages 231–244. IOS Press, 2010.

[12] M. Akram and K. H. Dar. On n-graphs. *Southeast Asian Bulletin of Mathematics*, 38:35–49, 2014.

[13] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[14] Muhammad Akram. Interval-valued fuzzy line graphs. *Neural Computing and Applications*, 21:145–150, 2012.

[15] Muhammad Akram, Arooj Adeel, and José Carlos Rodriguez Alcantud. Group decision-making methods based on hesitant n-soft sets. *Expert Syst. Appl.*, 115:95–105, 2019.

[16] Muhammad Akram, Arooj Adeel, and José Carlos Rodriguez Alcantud. Hesitant fuzzy n-soft sets: A new model with applications in decision-making. *J. Intell. Fuzzy Syst.*, 36:6113–6127, 2019.

[17] Muhammad Akram and Noura Omair Alshehri. Tempered interval-valued fuzzy hypergraphs. *University politehnica of bucharest scientific bulletin-series a-applied mathematics and physics*, 77(1):39–48, 2015.

[18] Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.

[19] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[20] Muhammad Akram, Feng Feng, Shahzad Sarwar, and Youne Bae Jun. Certain types of vague graphs. *University Politehnica of Bucharest Scientific Bulletin Series A*, 76(1):141–154, 2014.

[21] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[22] Muhammad Akram, MG Karunambigai, K Palanivel, and S Sivasankar. Balanced bipolar fuzzy graphs. *Journal of advanced research in pure mathematics*, 6(4):58–71, 2014.

[23] Muhammad Akram, Sheng-Gang Li, and KP Shum. Antipodal bipolar fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, 31(56):425–438, 2013.

[24] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[25] Muhammad Akram, Sundas Shahzadi, Areen Rasool, and Musavarah Sarwar. Decision-making methods based on fuzzy soft competition hypergraphs. *Complex & Intelligent Systems*, 8(3):2325–2348, 2022.

[26] Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.

[27] Muhammad Akram, Shumaiza, and José Carlos Rodríguez Alcantud. Multi-criteria decision making methods with bipolar fuzzy sets. *Forum for Interdisciplinary Mathematics*, 2023.

[28] Muhammad Akram, Shumaiza, and Florentin Smarandache. Decision-making with bipolar neutrosophic topsis and bipolar neutrosophic electre-i. *Axioms*, 7:33, 2018.

[29] Yousef Al-Qudah and Nasruddin Hassan. Operations on complex multi-fuzzy sets. *J. Intell. Fuzzy Syst.*, 33:1527–1540, 2017.

[30] Tareq M. Al-shami, Ljubisa D. R. Kocinac, and Baravan A. Asaad. Sum of soft topological spaces. *Mathematics*, 2020.

[31] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, Ali HM Al-Obaidi, et al. New concepts in partner multineutrosophic topological space. *International Journal of Neutrosophic Science*, (3):172–72, 2024.

[32] José Carlos R Alcantud, Azadeh Zahedi Khameneh, Gustavo Santos-García, and Muhammad Akram. A systematic literature review of soft set theory. *Neural Computing and Applications*, 36(16):8951–8975, 2024.

[33] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[34] Moslem Alimohammadlou and Zahra Khoshsepehr. The role of society 5.0 in achieving sustainable development: a spherical fuzzy set approach. *Environmental Science and Pollution Research*, 30:47630–47654, 2023.

[35] Abd Ulazeez M. Alkouri, Mo'men Bany Amer, and Ibrahim Jawarneh. Complex bipolar multi-fuzzy sets. *Uncertainty*, 2024.

[36] James Allen. *Natural language understanding*. Benjamin-Cummings Publishing Co., Inc., 1988.

[37] Marcos Antônio Alves, Petrônio Cândido de Lima e Silva, Carlos Alberto Severiano Junior, Gustavo Linhares Vieira, Frederico Gadelha Guimarães, and Hossein Javedani Sadaei. An extension of nonstationary fuzzy sets to heteroskedastic fuzzy time series. In *ESANN*, 2018.

[38] Ion Androutsopoulos, Graeme D Ritchie, and Peter Thanisch. Natural language interfaces to databases–an introduction. *Natural language engineering*, 1(1):29–81, 1995.

[39] Kenneth J. Arrow. Social choice and individual values. 1951.

[40] Baravan A Asaad, Sagvan Y Musa, and Zanyar A Ameen. Fuzzy bipolar hypersoft sets: A novel approach for decision-making applications. *Mathematical and Computational Applications*, 29(4):50, 2024.

[41] Shahzaib Ashraf, Saleem Abdullah, Tahir Mahmood, Fazal Ghani, and Tariq Mahmood. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst.*, 36:2829–2844, 2019.

[42] Muhammad Aslam. Introducing grubbs's test for detecting outliers under neutrosophic statistics - an application to medical data. *Journal of King Saud University - Science*, 2020.

[43] Muhammad Aslam. On detecting outliers in complex data using dixon's test under neutrosophic statistics. *Journal of King Saud University - Science*, 2020.

[44] Muhammad Aslam and Mohammed Albassam. Presenting post hoc multiple comparison tests under neutrosophic statistics. *Journal of King Saud University - Science*, 2020.

[45] Muhammad Aslam and Osama H. Arif. Testing of grouped product for the weibull distribution using neutrosophic statistics. *Symmetry*, 10:403, 2018.

[46] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[47] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[48] Krassimir T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[49] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[50] Krassimir T. Atanassov. On interval valued intuitionistic fuzzy sets. *Interval-Valued Intuitionistic Fuzzy Sets*, 2019.

[51] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[52] Krassimir T. Atanassov. New topological operator over intuitionistic fuzzy sets. 2023.

[53] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[54] Malik Shahzad Kaleem Awan and Mian Muhammad Awais. Predicting weather events using fuzzy rule based system. *Applied Soft Computing*, 11(1):56–63, 2011.

[55] Jawid Ahmad Baktash and Mursal Dawodi. Gpt-4: A review on advancements and opportunities in natural language processing. *arXiv preprint arXiv:2305.03195*, 2023.

[56] Jonathan D Barsky and Richard Labagh. A strategy for customer satisfaction. *Cornell Hotel and Restaurant Administration Quarterly*, 33(5):32–40, 1992.

[57] Rajamanickam Baskar, Kuo Ann Lee, Richard Yeo, and Kheng-Wei Yeoh. Cancer and radiation therapy: current advances and future directions. *International journal of medical sciences*, 9(3):193, 2012.

[58] Sadi Bayramov and Cigdem G. Aras. A new approach to separability and compactness in soft topological spaces. 2018.

[59] Michael J Behe. *Darwin's black box: The biochemical challenge to evolution*. Simon and Schuster, 1996.

[60] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[61] Anol Bhattacherjee. Understanding information systems continuance: An expectation-confirmation model. *MIS Q.*, 25:351–370, 2001.

[62] Monoranjan Bhowmik and Madhumangal Pal. *Intuitionistic neutrosophic set*. Infinite Study, 2009.

[63] Monoranjan Bhowmik and Madhumangal Pal. Intuitionistic neutrosophic set relations and some of its properties. *Journal of Information and Computing Science*, 5(3):183–192, 2010.

[64] S Bhuvaneshwari and C Antony Crispin Sweety. Neutrosophic spherical sets in mcdm. *Neutrosophic Sets and Systems*, 68:136–153, 2024.

[65] Steven Bird, Ewan Klein, and Edward Loper. *Natural language processing with Python: analyzing text with the natural language toolkit*. " O'Reilly Media, Inc.", 2009.

[66] Anamika Bisane, Shivanand Chandravanshi, Palash Manoj Thakre, Purab Kesharwani, and Atiya Khan. A comprehensive product review system for improved customer satisfaction. *2023 2nd International Conference on Paradigm Shifts in Communications Embedded Systems, Machine Learning and Signal Processing (PCEMS)*, pages 1–6, 2023.

[67] Isabelle Bloch. Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology. *Inf. Sci.*, 181:2002–2015, 2011.

[68] Isabelle Bloch. Mathematical morphology on bipolar fuzzy sets: general algebraic framework. *Int. J. Approx. Reason.*, 53:1031–1060, 2012.

[69] Borzoo Bonakdarpour and Sarai Sheinvald. Automata for hyperlanguages. *arXiv preprint arXiv:2002.09877*, 2020.

[70] Borzoo Bonakdarpour and Sarai Sheinvald. Finite-word hyperlanguages. *Information and Computation*, 295:104944, 2023.

[71] Gourangajit Borah and Palash Dutta. Fuzzy risk analysis in crop selection using information measures on quadripartitioned single-valued neutrosophic sets. *Expert Syst. Appl.*, 255:124750, 2024.

[72] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[73] Samuel R Bowman, Gabor Angeli, Christopher Potts, and Christopher D Manning. A large annotated corpus for learning natural language inference. *arXiv preprint arXiv:1508.05326*, 2015.

[74] George EP Box, J Stuart Hunter, William G Hunter, et al. Statistics for experimenters. In *Wiley series in probability and statistics*. Wiley Hoboken, NJ, 2005.

[75] George EP Box, William H Hunter, Stuart Hunter, et al. *Statistics for experimenters*, volume 664. John Wiley and sons New York, 1978.

[76] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[77] Marc Brisson and W. John Edmunds. Economic evaluation of vaccination programs: The impact of herd-immunity. *Medical Decision Making*, 23:76 – 82, 2003.

[78] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[79] Said Broumi. A single valued neutrosophic inventory model with neutrosophic random variable. *International Journal of Neutrosophic Science*, 2020.

[80] Said Broumi and Florentin Smarandache. Several similarity measures of neutrosophic sets. *Infinite Study*, 410(1), 2013.

[81] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[82] Richard Hubert Bruck. *A survey of binary systems*, volume 20. Springer, 1971.

[83] James J. Buckley. Fuzzy statistics: hypothesis testing. *Soft Computing*, 9:512–518, 2005.

[84] James J. Buckley. Fuzzy probability and statistics. 2006.

[85] M Burgin and A Krinik. Introduction to conditional hyperprobabilities. *Integration*, 2(3):285, 2011.

[86] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[87] Walter Alexandre Carnielli and Marcelo Esteban Coniglio. Paraconsistent set theory by predicating on consistency. *J. Log. Comput.*, 26:97–116, 2016.

[88] Susumu Cato. Quasi-decisiveness, quasi-ultrafilter, and social quasi-orderings. *Social Choice and Welfare*, 41(1):169–202, 2013.

[89] Alain Chateauneuf. Continuous representation of a preference relation on a connected topological space. *Journal of Mathematical Economics*, 16:139–146, 1987.

[90] Rajashi Chatterjee, Pinaki Majumdar, and Syamal Kumar Samanta. On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 30:2475–2485, 2015.

[91] AK Chaudhuri and P Das. Fuzzy connected sets in fuzzy topological spaces. *Fuzzy Sets and Systems*, 49(2):223–229, 1992.

[92] Chen-Tung Chen. Extensions of the topsis for group decision-making under fuzzy environment. *Fuzzy Sets Syst.*, 114:1–9, 2000.

[93] Jiqian Chen, Jun Ye, and Shigui Du. Scale effect and anisotropy analyzed for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. *Symmetry*, 9:208, 2017.

[94] Qian Chen, Xiaodan Zhu, Zhenhua Ling, Si Wei, Hui Jiang, and Diana Inkpen. Enhanced lstm for natural language inference. *arXiv preprint arXiv:1609.06038*, 2016.

[95] Shyi-Ming Chen. Measures of similarity between vague sets. *Fuzzy sets and Systems*, 74(2):217–223, 1995.

[96] Wei Chen, Wei Chen, Hamid Reza Pourghasemi, M. Panahi, Aiding Kornejady, Jiale Wang, Xiaoshen Xie, and Shubo Cao. Spatial prediction of landslide susceptibility using an adaptive neuro-fuzzy inference system combined with frequency ratio, generalized additive model, and support vector machine techniques. *Geomorphology*, 297:69–85, 2017.

[97] Daizhan Cheng, Xiaoming Hu, and Tielong Shen. Topological space.

[98] KR1442 Chowdhary and KR Chowdhary. Natural language processing. *Fundamentals of artificial intelligence*, pages 603–649, 2020.

[99] Victor Christianto. Remark on neutrosophic genetics: A generalized way towards five schemes of mutation and transmutation. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 42, 2021.

[100] Gilbert A Churchill Jr and Carol Surprenant. An investigation into the determinants of customer satisfaction. *Journal of marketing research*, 19(4):491–504, 1982.

[101] Ronan Collobert, Jason Weston, Léon Bottou, Michael Karlen, Koray Kavukcuoglu, and Pavel Kuksa. Natural language processing (almost) from scratch. *Journal of machine learning research*, 12:2493–2537, 2011.

[102] Alexis Conneau, Douwe Kiela, Holger Schwenk, Loïc Barrault, and Antoine Bordes. Supervised learning of universal sentence representations from natural language inference data. *arXiv preprint arXiv:1705.02364*, 2017.

[103] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[104] Bui Cong Cuong and Pham Van Hai. Some fuzzy logic operators for picture fuzzy sets. *2015 Seventh International Conference on Knowledge and Systems Engineering (KSE)*, pages 132–137, 2015.

[105] Rakhal Das. Spatial fuzzy topological space. *Proyecciones (Antofagasta)*, 2022.

[106] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[107] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[108] José R González de Mendívil. A generalization of myhill–nerode theorem for fuzzy languages. *Fuzzy Sets and Systems*, 301:103–115, 2016.

[109] José R González de Mendívil and Jose R Garitagoitia. Fuzzy languages with infinite range accepted by fuzzy automata: Pumping lemma and determinization procedure. *Fuzzy Sets and Systems*, 249:1–26, 2014.

[110] Aurore Delaigle and Peter Hall. Defining probability density for a distribution of random functions. *arXiv: Statistics Theory*, 2010.

[111] Irfan Deli. Refined neutrosophic sets and refined neutrosophic soft sets: theory and applications. In *Handbook of research on generalized and hybrid set structures and applications for soft computing*, pages 321–343. IGI Global, 2016.

[112] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[113] Irfan Deli, S Yusuf, Florentin Smarandache, and Mumtaz Ali. Interval valued bipolar neutrosophic sets and their application in pattern recognition. In *IEEE world congress on computational intelligence*, 2016.

[114] Nandini Dendukuri and Lawrence Joseph. Bayesian approaches to modeling the conditional dependence between multiple diagnostic tests. *Biometrics*, 57, 2001.

[115] G. Devika and Sudhaman Parthasarathy. Fuzzy statistics-based affinity propagation technique for clustering in satellite cloud image. *European Journal of Remote Sensing*, 51:754 – 764, 2018.

[116] Vincent T DeVita Jr and Edward Chu. A history of cancer chemotherapy. *Cancer research*, 68(21):8643–8653, 2008.

[117] Jacob Devlin. Bert: Pre-training of deep bidirectional transformers for language understanding. *arXiv preprint arXiv:1810.04805*, 2018.

[118] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[119] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[120] Kanghua Du and Yuming Du. Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:2813–2822, 2023.

[121] Sayipjamal Dulat, Sayipjamal Dulat, Tie-Jiun Hou, Jun Gao, Marco Guzzi, Joey Huston, Pavel M. Nadolsky, Jon Pumplin, Carl R. Schmidt, Daniel Robert Stump, and C.-P. Yuan. New parton distribution functions from a global analysis of quantum chromodynamics. *Physical Review D*, 93:033006, 2015.

[122] Ward Edwards. The theory of decision making. *Psychological bulletin*, 51(4):380, 1954.

[123] Kathleen M Eisenhardt and Mark J Zbaracki. Strategic decision making. *Strategic management journal*, 13(S2):17–37, 1992.

[124] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[125] Alvar Ellegård. *Darwin and the general reader: the reception of Darwin's theory of evolution in the British periodical press, 1859-1872*. University of Chicago Press, 1990.

[126] Ilijas Farah. Approximate homomorphisms. *Combinatorica*, 18:335–348, 1998.

[127] Nabilla Farah, Raissa Maharani, Novandra Rhezza Pratama, and M. Dachyar. E-commerce promotional products selection using swara and topsis. *International Journal of Innovative Science and Research Technology (IJISRT)*, 2024.

[128] Shuang Feng and C.L. Philip Chen. Fuzzy broad learning system: A novel neuro-fuzzy model for regression and classification. *IEEE Transactions on Cybernetics*, 50:414–424, 2020.

[129] Xuning Feng, Minggao Ouyang, Xiang Liu, Languang Lu, Yong Xia, and Xiangming He. Thermal runaway mechanism of lithium ion battery for electric vehicles: A review. *Energy Storage Materials*, 10:246–267, 2018.

[130] Abdolvahhab Fetanat, Mohsen Tayebi, and Mehran Moteraghi. Technology evaluation for biogas production from animal waste in circular carbon economy: A complex spherical fuzzy set-based decision-making framework. *Bioresource Technology Reports*, 2023.

[131] Aldo Figallo-Orellano and Juan Sebastian Slagter. Models for da costa's paraconsistent set theory. 2020.

[132] Hadar Frenkel and Sarai Sheinvald. Realizable and context-free hyperlanguages. *arXiv preprint arXiv:2209.10306*, 2022.

[133] Kevin Fuchs. Exploring the opportunities and challenges of nlp models in higher education: is chat gpt a blessing or a curse? In *Frontiers in Education*, volume 8, page 1166682. Frontiers Media SA, 2023.

[134] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 437.

[135] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. 2024.

[136] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[137] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[138] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. 2024.

[139] Takaaki Fujita. Review of some superhypergraph classes: Directed, bidirected, soft, and rough. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 86, 2024.

[140] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[141] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[142] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[143] Takaaki Fujita. A theoretical exploration of hyperconcepts: Hyperfunctions, hyperrandomness, hyperdecision-making, and beyond (including a survey of hyperstructures). 2024.

[144] Takaaki Fujita. Theoretical interpretations of large uncertain and hyper language models: Advancing natural uncertain and hyper language processing. 2024.

[145] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). 2024.

[146] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[147] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[148] Takaaki Fujita and Smarandache Florentin. Some graph parameters for superhypertree-width and neutrosophictree-width. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[149] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[150] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[151] Takaaki Fujita and Florentin Smarandache. Fundamental computational problems and algorithms for superhypergraphs. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[152] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[153] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[154] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[155] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[156] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[157] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[158] GA Ganati, VNS Rao Repalle, MA Ashebo, and M Amini. Turiyam graphs and its applications. *Information Sciences Letters*, 12(6):2423–2434, 2023.

[159] Abdul Haseeb Ganie, Surender Singh, and P. K. Bhatia. Some new correlation coefficients of picture fuzzy sets with applications. *Neural Computing and Applications*, 32:12609 – 12625, 2020.

[160] Rui Gao and Jianrong Wu. Filter with its applications in fuzzy soft topological spaces. *AIMS Mathematics*, 6(3):2359–2368, 2021.

[161] Jonathan M Garibaldi, Marcin Jaroszewski, and Salang Musikasuwan. New concepts related to non-stationary fuzzy sets. In *2007 IEEE International Fuzzy Systems Conference*, pages 1–6. IEEE, 2007.

[162] Audrey P. Gasch and Michael B. Eisen. Exploring the conditional coregulation of yeast gene expression through fuzzy k-means clustering. *Genome Biology*, 3:research0059.1 – research0059.22, 2002.

[163] James Paul Gee. Social linguistics and literacies: Ideology in discourse. 1996.

[164] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[165] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[166] Gerd Gigerenzer and Wolfgang Gaissmaier. Heuristic decision making. *Annual review of psychology*, 62(1):451–482, 2011.

[167] Allen S Ginsberg and Fred L Offensend. An application of decision theory to a medical diagnosis-treatment problem. *IEEE Transactions on Systems Science and Cybernetics*, 4(3):355–362, 1968.

[168] Yoav Goldberg. *Neural network methods in natural language processing*. Morgan & Claypool Publishers, 2017.

[169] S Gomathi, M Karpagadevi, S Krishnaprakash, A Revathy, and Said Broumi. Cubic spherical neutrosophic sets for advanced decision-making. *Neutrosophic Sets and Systems*, 73(1):24, 2024.

[170] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[171] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions and tractable queries. In *Proceedings of the eighteenth ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems*, pages 21–32, 1999.

[172] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[173] Georg Gottlob and Reinhard Pichler. Hypergraphs in model checking: Acyclicity and hypertree-width versus clique-width. *SIAM Journal on Computing*, 33(2):351–378, 2004.

[174] Carlos Granados. New notions on neutrosophic random variables. 2021.

[175] Carlos Granados and José Rubén Sanabria. On independence neutrosophic random variables. 2021.

[176] Peter R Grant. *Ecology and Evolution of Darwin's Finches (Princeton Science Library Edition)*. Princeton University Press, 2017.

[177] Peter R Grant and B Rosemary Grant. *40 Years of Evolution: Darwin's Finches on Daphne Major Island, New Edition*. Princeton University Press, 2024.

[178] Khara D. Grieger, Igor Linkov, Steffen Foss Hansen, and Anders Baun. Environmental risk analysis for nanomaterials: Review and evaluation of frameworks. *Nanotoxicology*, 6:196 – 212, 2012.

[179] Abhishek Guleria and Rakesh Kumar Bajaj. Eigen spherical fuzzy set and its application in decision making problem. *Scientia Iranica*, 2019.

[180] Fatma Kutlu Gündogdu and Cengiz Kahraman. A novel vikor method using spherical fuzzy sets and its application to warehouse site selection. *J. Intell. Fuzzy Syst.*, 37:1197–1211, 2019.

[181] Fatma Kutlu Gündogdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *J. Intell. Fuzzy Syst.*, 36:337–352, 2019.

[182] Yanhui Guo, Zelal Mustafaoglu, and Deepika Koundal. Spam detection using bidirectional transformers and machine learning classifier algorithms. 2023.

[183] Yanhui Guo and Abdulkadir Sengur. Ncm: Neutrosophic c-means clustering algorithm. *Pattern Recognition*, 48(8):2710–2724, 2015.

[184] Himanshu Gupta and Mukesh Kumar Barua. Supplier selection among smes on the basis of their green innovation ability using bwm and fuzzy topsis. *Journal of Cleaner Production*, 152:242–258, 2017.

[185] RK Gupta, Rama Mehta, Vijaya Agarwala, Bhanu Pant, and PP Sinha. Ductility prediction of ti aluminide intermetallics through neuro-fuzzy set approach. *Transactions of the Indian Institute of Metals*, 63:833–839, 2010.

[186] Mostafa Hajiaghaei-Keshteli, Zeynep Cenk, Babek Erdebilli, Yavuz Selim Özdemir, and Fatemeh Gholian-Jouybari. Pythagorean fuzzy topsis method for green supplier selection in the food industry. *Expert Syst. Appl.*, 224:120036, 2023.

[187] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[188] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[189] Walid Hariri. Unlocking the potential of chatgpt: A comprehensive exploration of its applications, advantages, limitations, and future directions in natural language processing. *arXiv preprint arXiv:2304.02017*, 2023.

[190] Michael A Harrison. *Introduction to formal language theory*. Addison-Wesley Longman Publishing Co., Inc., 1978.

[191] John A. Hartigan and M. Anthony. Wong. A k-means clustering algorithm. 1979.

[192] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[193] R. Hema, R. Sudharani, and M. Kavitha. A novel approach on plithogenic interval valued neutrosophic hypersoft sets and its application in decision making. *Indian Journal Of Science And Technology*, 2023.

[194] Gary G Hendrix, Earl D Sacerdoti, Daniel Sagalowicz, and Jonathan Slocum. Developing a natural language interface to complex data. *ACM Transactions on Database Systems (TODS)*, 3(2):105–147, 1978.

[195] Doeko Homan. Very basic set theory. 2023.

[196] Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.

[197] Haoyuan Hong, M. Panahi, Ataollah Shirzadi, Tianwu Ma, Junzhi Liu, A-Xing Zhu, Wei Chen, Ioannis Kougias, and Nerantzis Kazakis. Flood susceptibility assessment in hengfeng area coupling adaptive neuro-fuzzy inference system with genetic algorithm and differential evolution. *The Science of the total environment*, 621:1124–1141, 2018.

[198] John E Hopcroft and Jeffrey D Ullman. *Formal languages and their relation to automata*. Addison-Wesley Longman Publishing Co., Inc., 1969.

[199] Qingqing Hu and Xiaohong Zhang. Three-way fuzzy sets and their applications (iii). *Axioms*, 12:57, 2023.

[200] Han-Chen Huang and Xiao-Jun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, 24:213–228, 2016.

[201] Han-Chen Huang and Xiaojun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(02):213–227, 2016.

[202] S Satham Hussain, Isnaini Rosyida, Hossein Rashmanlou, and Farshid Mofidnakhaei. Interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis. *Computational and Applied Mathematics*, 40(4):121, 2021.

[203] Abiodun Motunrayo Ikotun, Ezugwu E. Absalom, Laith Mohammad Abualigah, Belal Abuhaija, and Heming Jia. K-means clustering algorithms: A comprehensive review, variants analysis, and advances in the era of big data. *Inf. Sci.*, 622:178–210, 2022.

[204] Maria Infusino. Topological vector spaces. 2018.

[205] Parthasarathy Iswarya and Dr. K. Bageerathi. On neutrosophic semi-open sets in neutrosophic topological spaces. *International Journal of Mathematics Trends and Technology*, 37:214–223, 2016.

[206] Muhammad Naveed Jafar, Muhammad Haris Saeed, Kainat Muniba Khan, Faten S. Alamri, and Hamiden A. Wahed Khalifa. Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. *IEEE Access*, PP:1–1, 2022.

[207] Sirus Jahanpanah and Roohallah Daneshpayeh. On derived superhyper be-algebras. *Neutrosophic Sets and Systems*, 57(1):21, 2023.

[208] Sirus Jahanpanah and Roohallah Daneshpayeh. An outspread on valued logic superhyperalgebras. *Facta Universitatis, Series: Mathematics and Informatics*, pages 427–437, 2024.

[209] Anil K. Jain. Data clustering: 50 years beyond k-means. *Pattern Recognit. Lett.*, 31:651–666, 2008.

[210] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[211] Ziwei Ji, Nayeon Lee, Rita Frieske, Tiezheng Yu, Dan Su, Yan Xu, Etsuko Ishii, Ye Jin Bang, Andrea Madotto, and Pascale Fung. Survey of hallucination in natural language generation. *ACM Computing Surveys*, 55(12):1–38, 2023.

[212] Wenxiang Jiao, Wenxuan Wang, Jen tse Huang, Xing Wang, and Zhaopeng Tu. Is chatgpt a good translator? yes with gpt-4 as the engine. 2023.

[213] Calvin Jongsma. Basic set theory and combinatorics. *Undergraduate Texts in Mathematics*, 2019.

[214] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[215] Young Bae Jun, Min Su Kang, and Seok Zun Song. Several types of bipolar fuzzy hyper bck-ideals in hyper bck-algebras. 2012.

[216] Young Bae Jun, Seon Jeong Kim, and Seok Zun Song. Hyper permeable values and energetic sets in bck/bci-algebras. 2020.

[217] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[218] Muhammad Junaid, Ye Xue, Muzzammil Wasim Syed, Ji Zu Li, and Muhammad Ziaullah. A neutrosophic ahp and topsis framework for supply chain risk assessment in automotive industry of pakistan. *Sustainability*, 2019.

[219] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[220] Ilanthenral Kandasamy, WB Vasantha, Jagan M Obbineni, and Florentin Smarandache. Sentiment analysis of tweets using refined neutrosophic sets. *Computers in Industry*, 115:103180, 2020.

[221] Abraham Kandel and William J. Byatt. Fuzzy sets, fuzzy algebra, and fuzzy statistics. *Proceedings of the IEEE*, 66:1619–1639, 1978.

[222] Tapas Kanungo, David M. Mount, Nathan S. Netanyahu, Christine D. Piatko, Ruth Silverman, and Angela Y. Wu. An efficient k-means clustering algorithm: Analysis and implementation. *IEEE Trans. Pattern Anal. Mach. Intell.*, 24:881–892, 2002.

[223] Faruk Karaaslan. *Neutrosophic soft sets with applications in decision making*. Infinite Study, 2014.

[224] Abdullah Kargın, Florentin Smarandache, and Memet Şahin. *New Type Hyper Groups, New Type SuperHyper Groups and Neutro-New Type SuperHyper Groups*. Infinite Study, 2023.

[225] M Kaviyarasu, Muhammad Aslam, Farkhanda Afzal, Maha Mohammed Saeed, Arif Mehmood, and Saeed Gul. The connectivity indices concept of neutrosophic graph and their application of computer network, highway system and transport network flow. *Scientific Reports*, 14(1):4891, 2024.

[226] John L. Kelley, Isaac Namioka, W. F. Donoghue, Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith. Linear topological spaces. 1966.

[227] Faezehossadat Khademi, Sayed Mohammadmehdi Jamal, Neela Deshpande, and Shreenivas N. Londhe. Predicting strength of recycled aggregate concrete using artificial neural network, adaptive neuro-fuzzy inference system and multiple linear regression. *International journal of sustainable built environment*, 5:355–369, 2016.

[228] Kabu Khadka and Soniya Maharjan. Customer satisfaction and customer loyalty. *Centria University of Applied Sciences Pietarsaari*, 1(10):58–64, 2017.

[229] H. E. Khalid, G. D. Gungor, and M. A. N. Zainal. Neutrosophic superhyper bi-topological spaces: Original notions and new insights. *Neutrosophic Sets and Systems*, 51:33–45, 2022.

[230] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[231] Huseyin Selcuk Kilic and Ahmet Selcuk Yalcin. Comparison of municipalities considering environmental sustainability via neutrosophic dematel based topsis. *Socio-economic Planning Sciences*, page 100827, 2020.

[232] Dan Klein and Christopher D Manning. Fast exact inference with a factored model for natural language parsing. *Advances in neural information processing systems*, 15, 2002.

[233] Puay Ping Koh, Yiik Diew Wong, and P. Chandrasekar. Safety evaluation of pedestrian behaviour and violations at signalised pedestrian crossings. *Safety Science*, 70:143–152, 2014.

[234] S Krishnaprakash, R Mariappan, and Said Broumi. Cubic spherical neutrosophic sets and selection of electric truck using cosine similarity measure. *Neutrosophic Sets and Systems*, 67(1):15, 2024.

[235] Cherukuri Aswani Kumar and Sunitha C. Srinivas. Concept lattice reduction using fuzzy k-means clustering. *Expert Syst. Appl.*, 37:2696–2704, 2010.

[236] Lakshya Kumar, Apurva Jindal, and Nagendra Rao Velaga. Financial risk assessment and modelling of ppp based indian highway infrastructure projects. *Transport Policy*, 62:2–11, 2017.

[237] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[238] M Lewis. Bart: Denoising sequence-to-sequence pre-training for natural language generation, translation, and comprehension. *arXiv preprint arXiv:1910.13461*, 2019.

[239] Chaojun Li, Kuo-Kuang Chang, and Shih-Ming Ou. Applying fuzzy statistics to evaluate and allocate resources in seven star grotto national park area. *IOP Conference Series: Materials Science and Engineering*, 1113, 2021.

[240] Dan Li, Jitender S. Deogun, William Spaulding, and Bill Shuart. Towards missing data imputation: A study of fuzzy k-means clustering method. In *Rough Sets and Current Trends in Computing*, 2004.

[241] Irene Li, Keen You, Xiangru Tang, Yujie Qiao, Lucas Huang, Chia-Chun Hsieh, Benjamin Rosand, and Dragomir R. Radev. Ehrkit: A python natural language processing toolkit for electronic health record texts. *ArXiv*, abs/2204.06604, 2022.

[242] Mark Junjie Li, Michael K. P. Ng, Yiu ming Cheung, and Joshua Zhexue Huang. Agglomerative fuzzy k-means clustering algorithm with selection of number of clusters. *IEEE Transactions on Knowledge and Data Engineering*, 20:1519–1534, 2008.

[243] Xia Li, Zhisheng Zhong, Jianlong Wu, Yibo Yang, Zhouchen Lin, and Hong Liu. Expectation-maximization attention networks for semantic segmentation. *2019 IEEE/CVF International Conference on Computer Vision (ICCV)*, pages 9166–9175, 2019.

[244] Xihua Li, Yun Luo, Hongya Wang, Jiong Lin, and Bin Deng. Doctor selection based on aspect-based sentiment analysis and neutrosophic topsis method. *Eng. Appl. Artif. Intell.*, 124:106599, 2023.

[245] Yanfeng Li, Hongzhong Huang, Jinhua Mi, Weiwen Peng, and X. M. Han. Reliability analysis of multi-state systems with common cause failures based on bayesian network and fuzzy probability. *Annals of Operations Research*, 311:195 – 209, 2019.

[246] Zheng Li, Yue Zhao, Xiyang Hu, Nicola Botta, Cezar Ionescu, and George H. Chen. Ecod: Unsupervised outlier detection using empirical cumulative distribution functions. *IEEE Transactions on Knowledge and Data Engineering*, 35:12181–12193, 2022.

[247] Thierry Libert. Models for a paraconsistent set theory. *J. Appl. Log.*, 3:15–41, 2005.

[248] Xiao Liu, Yanan Zheng, Zhengxiao Du, Ming Ding, Yujie Qian, Zhilin Yang, and Jie Tang. Gpt understands, too. *AI Open*, 5:208–215, 2024.

[249] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[250] Languang Lu, Xuebing Han, Jianqiu Li, Jianfeng Hua, and Minggao Ouyang. A review on the key issues for lithium-ion battery management in electric vehicles. *Journal of Power Sources*, 226:272–288, 2013.

[251] Minxia Luo and Guofeng Zhang. Divergence-based distance for picture fuzzy sets and its application to multi-attribute decision-making. *Soft Computing*, pages 1–17, 2023.

[252] M. M., M. Suneetha, Maria Mikhailova, Sripada Nsvsc Ramesh, and Kollati Vijaya Kumar. Leveraging double-valued neutrosophic set for real-time chronic kidney disease detection and classification. *International Journal of Neutrosophic Science*.

[253] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.

[254] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[255] Pabitra Kumar Maji. *Neutrosophic soft set*. Infinite Study, 2013.

[256] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[257] Christopher D Manning. *Foundations of statistical natural language processing*. The MIT Press, 1999.

[258] Christopher D Manning, Mihai Surdeanu, John Bauer, Jenny Rose Finkel, Steven Bethard, and David McClosky. The stanford corenlp natural language processing toolkit. In *Proceedings of 52nd annual meeting of the association for computational linguistics: system demonstrations*, pages 55–60, 2014.

[259] James G March. *A primer on decision making: How decisions happen*. Simon and Schuster, 1994.

[260] Nivetha Martin. Plithogenic swara-topsis decision making on food processing methods with different normalization techniques. *Advances in Decision Making*, 69, 2022.

[261] Mourad Oqla Massa'deh. On hyper q-fuzzy normal hx subgroup, conjugate and its.

[262] Manoj Mathew, Ripon Kumar Chakrabortty, and Michael J. Ryan. A novel approach integrating ahp and topsis under spherical fuzzy sets for advanced manufacturing system selection. *Eng. Appl. Artif. Intell.*, 96:103988, 2020.

[263] John T. Mentzer, William DeWitt, James S. Keebler, Soon Hong Min, Nancy W. Nix, Carlo D. Smith, and Zach G. Zacharia. Defining supply chain management. *Journal of Business Logistics*, 22:1–25, 2001.

[264] Irwin Miller. Probability, random variables, and stochastic processes. *Technometrics*, 8:378–380, 1966.

[265] Geltrude Mingrone, Simona Panunzi, Andrea De Gaetano, Caterina Guidone, Amerigo Iaconelli, Laura Leccesi, Giuseppe Nanni, Alfons Pomp, Marco Castagneto, Giovanni Ghirlanda, et al. Bariatric surgery versus conventional medical therapy for type 2 diabetes. *New England Journal of Medicine*, 366(17):1577–1585, 2012.

[266] Md Mofijur, I. M. Rizwanul Fattah, Md. Asraful Alam, A.K.M. Saiful Islam, Hwai Chyuan Ong, S. M. Ashrafur Rahman, Golamhassan Najafi, S.F. Ahmed, Md. Alhaz Uddin, and Teuku Meurah Indra Mahlia. Impact of covid-19 on the social, economic, environmental and energy domains: Lessons learnt from a global pandemic. *Sustainable Production and Consumption*, 26:343 – 359, 2020.

[267] G Mohanapriya, S Muthukumar, MM Shanmugapriya, et al. Kalman bucy filtered neuro fuzzy image denoising for medical image processing. *Neutrosophic Sets and Systems*, 70:314–330, 2024.

[268] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[269] Bernard Monjardet. On the use of ultrafilters in social choice theory. *Social choice and welfare*, 5:73–78, 1983.

[270] Sagvan Y Musa and Baravan A Asaad. A novel approach towards parameter reduction based on bipolar hypersoft set and its application to decision-making. *Neutrosophic Sets and Systems*, 55(1):33, 2023.

[271] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[272] Nada A Nabeeh, Florentin Smarandache, Mohamed Abdel-Basset, Haitham A El-Ghareeb, and Ahmed Aboelfetouh. An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. *Ieee Access*, 7:29734–29744, 2019.

[273] Nada A. Nabeeh, Florentin Smarandache, Mohamed Abdel-Basset, Haitham A. El-Ghareeb, and Ahmed Aboelfetouh. An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. *IEEE Access*, 7:29734–29744, 2019.

[274] Sorin Nădăban, Simona Dzitac, and Ioan Dzitac. Fuzzy topsis: a general view. *Procedia computer science*, 91:823–831, 2016.

[275] V Nayagam et al. A total ordering on n-valued refined neutrosophic sets using dictionary ranking based on total ordering on n-valued neutrosophic tuplets. *Neutrosophic Sets and Systems*, 58(1):23, 2023.

[276] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[277] Zohreh Nazari and Batool Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16:173 – 185, 2018.

[278] Constantin Virgil Negoita and Anca L. Ralescu. Simulation, knowledge-based computing, and fuzzy statistics. *Technometrics*, 31:129–130, 1987.

[279] Gia Nhu Nguyen, Le Hoang Son, Amira S Ashour, and Nilanjan Dey. A survey of the state-of-the-arts on neutrosophic sets in biomedical diagnoses. *International Journal of Machine Learning and Cybernetics*, 10:1–13, 2019.

[280] Robert W. Noyes, Arthur T. Hertig, and John Andrew Rock. Dating the endometrial biopsy. *American journal of obstetrics and gynecology*, 122 2:262–3, 1950.

[281] Serafim Opricovic and Gwo-Hshiung Tzeng. Compromise solution by mcdm methods: A comparative analysis of vikor and topsis. *Eur. J. Oper. Res.*, 156:445–455, 2004.

[282] Krzysztof Palczewski and Wojciech Sałabun. The fuzzy topsis applications in the last decade. *Procedia Computer Science*, 159:2294–2303, 2019.

[283] Dragan Pamucar, Morteza Yazdani, Radojko Obradovic, Anil Kumar, and Mercedes Torres-Jiménez. A novel fuzzy hybrid neutrosophic decision-making approach for the resilient supplier selection problem. *International Journal of Intelligent Systems*, 35(12):1934–1986, 2020.

[284] Athanasios Papoulis. Probability, random variables and stochastic processes. 1965.

[285] Athanasios Papoulis and S. Unnikrishna Pillai. Probability, random variables, and stochastic processes. 2002.

[286] Antonios Paraskevas and Michael Madas. A neutrosophic model for measuring evolution, involution, and indeterminacy in species: Integrating common and uncommon traits in environmental adaptation. *Neutrosophic Systems with Applications*, 23:23–32, 2024.

[287] Antonios Paraskevas and Florentin Smarandache. Modeling social evolution, involution, and indeterminacy: A neutrosophic-cultural algorithm approach. *Neutrosophic Sets and Systems*, 77:153–168, 2025.

[288] Vasile Patrascu. Penta and hexa valued representation of neutrosophic information. *arXiv preprint arXiv:1603.03729*, 2016.

[289] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[290] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[291] Zdzisław Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[292] Zdzisław Pawlak. *Rough sets: Theoretical aspects of reasoning about data*, volume 9. Springer Science & Business Media, 2012.

[293] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[294] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[295] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[296] Xindong Peng and Yong Yang. Some results for pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 30, 2015.

[297] Zhang peng Tian, Hong yu Zhang, Jing Wang, Jian qiang Wang, and Xiao hong Chen. Multi-criteria decision-making method based on a cross-entropy with interval neutrosophic sets. *International Journal of Systems Science*, 47:3598 – 3608, 2015.

[298] Arthur E Poropat. A meta-analysis of the five-factor model of personality and academic performance. *Psychological bulletin*, 135(2):322, 2009.

[299] John G. Proakis. Probability, random variables and stochastic processes. *IEEE Trans. Acoust. Speech Signal Process.*, 33:1637, 1985.

[300] Jhp Julwan Hendry Purba. Fuzzy probability on reliability study of nuclear power plant probabilistic safety assessment: A review. *Progress in Nuclear Energy*, 76:73–80, 2014.

[301] Peng Qi, Yuhao Zhang, Yuhui Zhang, Jason Bolton, and Christopher D. Manning. Stanza: A python natural language processing toolkit for many human languages. *ArXiv*, abs/2003.07082, 2020.

[302] Mohammad Naved Qureshi and Mohd Vasim Ahamad. An improved method for image segmentation using k-means clustering with neutrosophic logic. *Procedia computer science*, 132:534–540, 2018.

[303] Alec Radford. Improving language understanding by generative pre-training. 2018.

[304] Atiqe Ur Rahman, Muhammad Haris Saeed, and Hamiden Abd El-Wahed Khalifa. Multi-attribute decision-making based on aggregations and similarity measures of neutrosophic hypersoft sets with possibility setting. *Journal of Experimental & Theoretical Artificial Intelligence*, 36:161 – 186, 2022.

[305] George Rahonis. Fuzzy languages. In *Handbook of Weighted Automata*, pages 481–517. Springer, 2009.

[306] D. Rajan. Probability, random variables, and stochastic processes. 2017.

[307] Hao ran Lin, Bing yuan Cao, and Yun zhang Liao. Fuzzy statistics and fuzzy probability. 2018.

[308] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[309] M Murali Krishna Rao. Tripolar fuzzy interior ideals and tripolar fuzzy soft interior ideals over semigroups. *Annals of Fuzzy Mathematics and Informatics*, 20(3):243–256, 2020.

[310] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[311] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, (422019):731743, 2019.

[312] V. Venkateswara Rao and Y. Srinivasa Rao. Neutrosophic pre-open sets and pre-closed sets in neutrosophic topology. *viXra*, 2017.

[313] Abdolreza Rashno, Florentin Smarandache, and Saeed Sadri. Refined neutrosophic sets in content-based image retrieval application. In *2017 10th Iranian Conference on Machine Vision and Image Processing (MVIP)*, pages 197–202. IEEE, 2017.

[314] Lisbeth J Reales-Chacón, María E Lucena de Ustáriz, Francisco J Ustáriz-Fajardo, Andrea C Peñafiel Luna, Gabriela J Bonilla-Ayala, Pablo Djabayan-Djibeyan, José L Erazo-Parra, and Mónica A Valdiviezo-Maygua. Study of the efficacy of neural mobilizations to improve sensory and functional responses of lower extremities in older adults with diabetic peripheral neuropathy using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 74:1–12, 2024.

[315] Stefano Crespi Reghizzi, Luca Breveglieri, and Angelo Morzenti. *Formal languages and compilation*. Springer, 2013.

[316] Ehud Reiter and Robert Dale. Building applied natural language generation systems. *Natural Language Engineering*, 3(1):57–87, 1997.

[317] Akbar Rezaei, Tahsin Oner, Tugce Katican, Florentin Smarandache, and N Gandotra. *A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets*. Infinite Study, 2022.

[318] Muhammad Riaz and Syeda Tayyba Tehrim. A robust extension of vikor method for bipolar fuzzy sets using connection numbers of spa theory based metric spaces. *Artificial Intelligence Review*, 54:561 – 591, 2020.

[319] Jesús Estupiñán Ricardo, Jorge Juan Domínguez Menéndez, Ignacio Fernando Barcos Arias, Jorge Manuel Macías Bermúdez, and Noel Moreno Lemus. Neutrosophic k-means for the analysis of earthquake data in ecuador. *Neutrosophic Sets & Systems*, 44, 2021.

[320] Robert J Richards. *The meaning of evolution: The morphological construction and ideological reconstruction of Darwin's theory*. University of Chicago Press, 2009.

[321] Michelle Richardson, Charles Abraham, and Rod Bond. Psychological correlates of university students' academic performance: a systematic review and meta-analysis. *Psychological bulletin*, 138(2):353, 2012.

[322] John T Rickard, Janet Aisbett, and Greg Gibbon. Fuzzy subsethood for fuzzy sets of type-2 and generalized type-$\{n\}$. *IEEE Transactions on Fuzzy Systems*, 17(1):50–60, 2008.

[323] Denis Riordan and Bjarne K Hansen. A fuzzy case-based system for weather prediction. *Engineering Intelligent Systems for Electrical Engineering and Communications*, 10(3):139–146, 2002.

[324] Rosa M Rodríguez, Luis Martínez, Vicenç Torra, ZS Xu, and Francisco Herrera. Hesitant fuzzy sets: state of the art and future directions. *International journal of intelligent systems*, 29(6):495–524, 2014.

[325] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[326] P Roopadevi, M Karpagadevi, S Krishnaprakash, Said Broumi, and S Gomathi. Comprehensive decision-making with spherical fermatean neutrosophic sets in structural engineering. *International Journal of Neutrosophic Science (IJNS)*, 24(4), 2024.

[327] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[328] Benjamin Rossman. Homomorphism preservation theorems. *J. ACM*, 55:15:1–15:53, 2008.

[329] Grzegorz Rozenberg and Arto Salomaa. *Handbook of Formal Languages: Volume 3 Beyond Words*. Springer Science & Business Media, 2012.

[330] Farnaz Sabahi and Mohammad Reza Akbarzadeh-Totonchi. Introducing validity in fuzzy probability for judicial decision-making. *Int. J. Approx. Reason.*, 55:1383–1403, 2014.

[331] Abhijit Saha, Irfan Deli, and Said Broumi. Hesitant triangular neutrosophic numbers and their applications to madm. *Neutrosophic Sets and Systems*, 35:269–298, 2020.

[332] Mehmet Sahin, Vakkas Ulucay, and Merve Meneke. Some new operations of (α,β,γ) interval cut set of interval valued neutrosophic sets. *Journal of Mathematical and Fundamental Sciences*, 2018.

[333] Rıdvan Şahin and Ahmet Küçük. On similarity and entropy of neutrosophic soft sets. *Journal of intelligent & fuzzy systems*, 27(5):2417–2430, 2014.

[334] Kavita Sahu, Fahad Ahmed Al-Zahrani, R. K. Srivastava, and Rajeev Kumar. Hesitant fuzzy sets based symmetrical model of decision-making for estimating the durability of web application. *Symmetry*, 12:1770, 2020.

[335] Rekha Sahu, Satya Ranjan Dash, and Sujit Das. Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory. 2021.

[336] A. A. Salama, Said Broumi, and Florentin Smarandache. Neutrosophic crisp open set and neutrosophic crisp continuity via neutrosophic crisp ideals. *International Journal of Information Engineering and Electronic Business*, 6:1–8, 2014.

[337] Mahmood M Salih, BB Zaidan, AA Zaidan, and Mohamed A Ahmed. Survey on fuzzy topsis state-of-the-art between 2007 and 2017. *Computers & Operations Research*, 104:207–227, 2019.

[338] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.

[339] Sabu Sebastian and TV Ramakrishnan. Multi-fuzzy sets. In *International Mathematical Forum*, volume 5, pages 2471–2476, 2010.

[340] Sabu Sebastian and TV Ramakrishnan. Multi-fuzzy sets: An extension of fuzzy sets. *Fuzzy Information and Engineering*, 3:35–43, 2011.

[341] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[342] Kristina P. Sinaga and Miin-Shen Yang. Unsupervised k-means clustering algorithm. *IEEE Access*, 8:80716–80727, 2020.

[343] Prem Kumar Singh. Turiyam set a fourth dimension data representation. *Journal of Applied Mathematics and Physics*, 9(7):1821–1828, 2021.

[344] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Sciences*, 18(1):57–72, 2022.

[345] Pritpal Singh and Surya Sekhar Bose. Ambiguous d-means fusion clustering algorithm based on ambiguous set theory: Special application in clustering of ct scan images of covid-19. *Knowledge-Based Systems*, 231:107432, 2021.

[346] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[347] Pushpinder Singh. Correlation coefficients for picture fuzzy sets. *J. Intell. Fuzzy Syst.*, 28:591–604, 2015.

[348] Joy K. Singhv and DeeptiKakkar. Source localization based healthcare diagnostic system using erp signals: An application to asd. 2021.

[349] Hans Slabbekoorn. Songs of the city: noise-dependent spectral plasticity in the acoustic phenotype of urban birds. *Animal Behaviour*, 85:1089–1099, 2013.

[350] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[351] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[352] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[353] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[354] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[355] Florentin Smarandache. *Definitions derived from neutrosophics*. Infinite Study, 2003.

[356] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[357] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[358] Florentin Smarandache. *Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability*. Infinite Study, 2013.

[359] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[360] Florentin Smarandache. Introduction to neutrosophic statistics. *ArXiv*, abs/1406.2000, 2014.

[361] Florentin Smarandache. Degrees of membership> 1 and< 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[362] Florentin Smarandache. Neutrosophic overset. *Neutrosophic Underset, and Neutrosophic Offset: Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics, Pons Editions Brussels*, 2016.

[363] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[364] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[365] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[366] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[367] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[368] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[369] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *ArXiv*, abs/1808.03948, 2018.

[370] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[371] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[372] Florentin Smarandache. *Introduction to neutrosophic genetics*. Infinite Study, 2020.

[373] Florentin Smarandache. *NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)*, volume 5. Infinite Study, 2021.

[374] Florentin Smarandache. Plithogenic probability & statistics are generalizations of multivariate probability & statistics. 2021.

[375] Florentin Smarandache. History of superhyperalgebra and neutrosophic superhyperalgebra (revisited again). *Neutrosophic Algebraic Structures and Their Applications*, page 10, 2022.

[376] Florentin Smarandache. Introducing a theory of neutrosophic evolution: Degrees of evolution, indeterminacy, and involution. *Collected Papers. Volume XIV: Neutrosophics and other topics*, page 159, 2022.

[377] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[378] Florentin Smarandache. Neutrosophic statistics is an extension of interval statistics, while plithogenic statistics is the most general form of statistics (second version). *International Journal of Neutrosophic Science*, 2022.

[379] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[380] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[381] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[382] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[383] Florentin Smarandache. New types of topologies and neutrosophic topologies. *Neutrosophic Systems with Applications*, 1:1–3, 2023.

[384] Florentin Smarandache. New types of topologies and neutrosophic topologies (improved version). *Neutrosophic Sets and Systems*, 57(1):14, 2023.

[385] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[386] Florentin Smarandache. Foundation of revolutionary topologies: An overview, examples, trend analysis, research issues, challenges, and future directions. 2024.

[387] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[388] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[389] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[390] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group. *Neural Computing and Applications*, 29(7):595–601, 2018.

[391] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[392] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[393] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. Applications of bimatrices to some fuzzy and neutrosophic models. 2005.

[394] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[395] Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay, and Abdullah Kargın. *Neutrosophic SuperHyperAlgebra and New Types of Topologies*. Infinite Study, 2023.

[396] Florentin Smarandache and AA Salama. Neutrosophic crisp set theory. 2015.

[397] Eduardo Soares, Pyramo Costa Jr, Bruno Costa, and Daniel Leite. Ensemble of evolving data clouds and fuzzy models for weather time series prediction. *Applied Soft Computing*, 64:445–453, 2018.

[398] Le Hoang Son, Pham Van Viet, and Pham Van Hai. Picture inference system: a new fuzzy inference system on picture fuzzy set. *Applied Intelligence*, 46:652 – 669, 2016.

[399] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[400] Samir K. Srivastava. Green supply-chain management: A state-of-the-art literature review. *IO: Regulation*, 2007.

[401] Donald E Stanley and Daniel G Campos. The logic of medical diagnosis. *Perspectives in Biology and Medicine*, 56(2):300–315, 2013.

[402] V. S. Subha and R. Selvakumar. An innovative electric vehicle selection with multi-criteria decision-making approach in indian brands on a neutrosophic hypersoft rough set by using reduct and core. *2024*

Fourth International Conference on Advances in Electrical, Computing, Communication and Sustainable Technologies (ICAECT), pages 1–7, 2024.

[403] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[404] Siti Noraini Sulaiman and Nor Ashidi Mat Isa. Adaptive fuzzy-k-means clustering algorithm for image segmentation. *IEEE Transactions on Consumer Electronics*, 56, 2010.

[405] Chia-Chi Sun. A performance evaluation model by integrating fuzzy ahp and fuzzy topsis methods. *Expert systems with applications*, 37(12):7745–7754, 2010.

[406] Heng Sun and Wang Ni. Design and application of an ai-based text content moderation system. *Scientific Programming*, 2022.

[407] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[408] Seyed Mahmoud Taheri. Trends in fuzzy statistics. *Austrian Journal of Statistics*, 32:239–257, 2016.

[409] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[410] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[411] Valery V. Tuchin. Tissue optics: Light scattering methods and instruments for medical diagnosis. 2000.

[412] Vakkas Uluçay. Some concepts on interval-valued refined neutrosophic sets and their applications. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7857–7872, 2021.

[413] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[414] Vakkas Ulucay, Adil Kılıç, Memet Sahin, and Harun Deniz. *A new hybrid distance-based similarity measure for refined neutrosophic sets and its application in medical diagnosis*. Infinite Study, 2019.

[415] Robert L Vaught. *Set theory: an introduction*. Springer Science & Business Media, 2001.

[416] Nikolai Konstantinovich Vereshchagin and Alexander Shen. *Basic set theory*. Number 17. American Mathematical Soc., 2002.

[417] Dhanasingh Sivalinga Vijayan, Eugeniusz Koda, Arvindan Sivasuriyan, Johannes Winkler, Parthiban Devarajan, Ramamoorthy Sanjay Kumar, Aleksandra Jakimiuk, Piotr Osiski, Anna Podlasek, and Magdalena Daria Vaverková. Advancements in solar panel technology in civil engineering for revolutionizing renewable energy solutions-a review. *Energies*, 2023.

[418] Chia-Nan Wang, Ngoc-Ai-Thy Nguyen, and Thanh-Tuan Dang. Offshore wind power station (owps) site selection using a two-stage mcdm-based spherical fuzzy set approach. *Scientific Reports*, 12, 2022.

[419] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[420] Jingqian Wang, Xiaohong Zhang, and Qingqing Hu. Three-way fuzzy sets and their applications (ii). *Axioms*, 11(10):532, 2022.

[421] Meng Wang and Fanghui Hu. The application of nltk library for python natural language processing in corpus research. *Theory and Practice in Language Studies*, 2021.

[422] Tao Wang, Xinxing Wu, Harish Garg, Li-Yu Daisy Liu, and Guanrong Chen. A prospect theory-based mabac algorithm with novel similarity measures and interactional operations for picture fuzzy sets and its applications. *Eng. Appl. Artif. Intell.*, 126:106787, 2023.

[423] Zhongjie Wang and Minghe Lin. Finite element analysis method of slope stability based on fuzzy statistics. *Earth Sciences Research Journal*, 25:123–130, 2021.

[424] Zach Weber. Transfinite numbers in paraconsistent set theory. *The Review of Symbolic Logic*, 3:71 – 92, 2010.

[425] Zach Weber. Transfinite cardinals in paraconsistent set theory. *The Review of Symbolic Logic*, 5:269 – 293, 2012.

[426] Guiwu Wei and Hui Gao. The generalized dice similarity measures for picture fuzzy sets and their applications. *Informatica*, 29:107–124, 2018.

[427] Guiwu Wei and Yu Wei. Similarity measures of pythagorean fuzzy sets based on the cosine function and their applications. *International Journal of Intelligent Systems*, 33:634 – 652, 2018.

[428] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[429] Samuel Stanley Wilks. *Mathematical statistics*. Read Books Ltd, 2012.

[430] David Wilson. *Darwin's cathedral: Evolution, religion, and the nature of society*. University of Chicago press, 2019.

[431] Kim Witte and Mike Allen. A meta-analysis of fear appeals: Implications for effective public health campaigns. *Health Education & Behavior*, 27:591 – 615, 2000.

[432] Thomas Wolf, Lysandre Debut, Victor Sanh, Julien Chaumond, Clement Delangue, Anthony Moi, Pierric Cistac, Tim Rault, Rémi Louf, Morgan Funtowicz, et al. Transformers: State-of-the-art natural language processing. In *Proceedings of the 2020 conference on empirical methods in natural language processing: system demonstrations*, pages 38–45, 2020.

[433] Paul M Wortman. Medical diagnosis: An information-processing approach. *Computers and Biomedical Research*, 5(4):315–328, 1972.

[434] Weizhi Wu and Lei Zhou. On intuitionistic fuzzy topologies based on intuitionistic fuzzy reflexive and transitive relations. *Soft Computing*, 15:1183–1194, 2011.

[435] Yonghui Wu, Mike Schuster, Z. Chen, Quoc V. Le, Mohammad Norouzi, Wolfgang Macherey, Maxim Krikun, Yuan Cao, Qin Gao, Klaus Macherey, Jeff Klingner, Apurva Shah, Melvin Johnson, Xiaobing Liu, Lukasz Kaiser, Stephan Gouws, Yoshikiyo Kato, Taku Kudo, Hideto Kazawa, Keith Stevens, George Kurian, Nishant Patil, Wei Wang, Cliff Young, Jason R. Smith, Jason Riesa, Alex Rudnick, Oriol Vinyals, Gregory S. Corrado, Macduff Hughes, and Jeffrey Dean. Google's neural machine translation system: Bridging the gap between human and machine translation. *ArXiv*, abs/1609.08144, 2016.

[436] Jinglin Xu, Junwei Han, Kai Xiong, and Feiping Nie. Robust and sparse fuzzy k-means clustering. In *International Joint Conference on Artificial Intelligence*, 2016.

[437] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.

[438] Rui Yang, Qingcheng Zeng, Keen You, Yujie Qiao, Lucas Huang, Chia-Chun Hsieh, Benjamin Rosand, Jeremy Goldwasser, Amisha D. Dave, Tiarnán D. L. Keenan, Emily Y. Chew, Dragomir R. Radev, Zhiyong Lu, Hua Xu, Qingyu Chen, and Irene Li. Ascle: A python natural language processing toolkit for medical text generation. 2023.

[439] Xibei Yang, Dongjun Yu, Jingyu Yang, and Chen Wu. Generalization of soft set theory: from crisp to fuzzy case. In *Fuzzy Information and Engineering: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE)*, pages 345–354. Springer, 2007.

[440] Yong Yang, Xia Tan, and Congcong Meng. The multi-fuzzy soft set and its application in decision making. *Applied Mathematical Modelling*, 37(7):4915–4923, 2013.

[441] Zhilin Yang. Xlnet: Generalized autoregressive pretraining for language understanding. *arXiv preprint arXiv:1906.08237*, 2019.

[442] Jun Ye and Shigui Du. Some distances, similarity and entropy measures for interval-valued neutrosophic sets and their relationship. *International Journal of Machine Learning and Cybernetics*, 10:347 – 355, 2017.

[443] Hasan Yeti? and Mehmet Karakose. Modelling type-2 fuzzy systems by optimized nonstationary fuzzy sets with genetic algorithm. *2020 24th International Conference on Information Technology (IT)*, pages 1–4, 2020.

[444] Jeonghee Yi, Tetsuya Nasukawa, Razvan Bunescu, and Wayne Niblack. Sentiment analyzer: Extracting sentiments about a given topic using natural language processing techniques. In *Third IEEE international conference on data mining*, pages 427–434. IEEE, 2003.

[445] Adem Yolcu and Taha Yasin Ozturk. Fuzzy hypersoft sets and it's application to decision-making. *Theory and application of hypersoft set*, 50, 2021.

[446] Keun-Young Yoo. Cancer control activities in the republic of korea. *Japanese journal of clinical oncology*, 38 5:327–33, 2008.

[447] Hong yu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615–627, 2016.

[448] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[449] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[450] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[451] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[452] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[453] Lemnaouar Zedam, Naeem Jan, Ewa Rak, Tahir Mahmood, and Kifayat Ullah. An approach towards decision-making and shortest path problems based on t-spherical fuzzy information. *International Journal of Fuzzy Systems*, 22:1521 – 1534, 2020.

[454] Aiwu Zhang, Jianguo Du, and Hongjun Guan. Interval valued neutrosophic sets and multi-attribute decision-making based on generalized weighted aggregation operator. *J. Intell. Fuzzy Syst.*, 29:2697–2706, 2015.

[455] Dayong Zhang, Minwen Hu, and Qiang Ji. Financial markets under the global pandemic of covid-19. *Finance Research Letters*, 36:101528 – 101528, 2020.

[456] Hongyu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615 – 627, 2015.

[457] Qi Zhang, Chunjie Zhou, Yu-Chu Tian, Naixue N. Xiong, Yuanqing Qin, and Bowen Hu. A fuzzy probability bayesian network approach for dynamic cybersecurity risk assessment in industrial control systems. *IEEE Transactions on Industrial Informatics*, 14:2497–2506, 2018.

[458] Qian-Sheng Zhang and Sheng-Yi Jiang. A note on information entropy measures for vague sets and its applications. *Information Sciences*, 178(21):4184–4191, 2008.

[459] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[460] Wen-Ran Zhang. Bipolar fuzzy sets. 1997.

[461] Wen-Ran Zhang. Yinyang bipolar fuzzy sets and fuzzy equilibrium relations: for clustering, optimization, and global regulation. *Int. J. Inf. Technol. Decis. Mak.*, 5:19–46, 2006.

[462] Xiaolu Zhang and Zeshui Xu. Extension of topsis to multiple criteria decision making with pythagorean fuzzy sets. *International Journal of Intelligent Systems*, 29, 2014.

[463] Hu Zhao and Hong-Ying Zhang. On hesitant neutrosophic rough set over two universes and its application. *Artificial Intelligence Review*, 53:4387–4406, 2020.

[464] Xuerong Zhao and Bao Qing Hu. Fuzzy and interval-valued fuzzy decision-theoretic rough set approaches based on fuzzy probability measure. *Inf. Sci.*, 298:534–554, 2015.

[465] H-J Zimmermann. Fuzzy set theory and mathematical programming. *Fuzzy sets theory and applications*, pages 99–114, 1986.

Chapter 2

Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets II

Takaaki Fujita[1]*, Florentin Smarandache[2]

[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp

[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract. This paper delves into the advancements of classical set theory to address the complexities and uncertainties inherent in real-world phenomena. It highlights three major extensions of traditional set theory—Fuzzy Sets [288], Neutrosophic Sets [237], and Plithogenic Sets [243]—and examines their further generalizations into Hyperfuzzy [106], HyperNeutrosophic [90], and Hyperplithogenic Sets [90].

Building on previous research [83], this study explores the potential applications of HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets across various domains. Specifically, it extends fundamental concepts such as Neutrosophic Logic, Cognitive Maps, Graph Neural Networks, Classifiers, and Triplet Groups through these advanced set structures and briefly analyzes their mathematical properties.

Keywords: Fuzzy set, Neutrosophic set, Hyperstructure, Hyperfuzzy set, Hyperneutrosophic set

---

1. Introduction

This paper is closely related to [83]. Readers are encouraged to review [83] in advance, as needed.

1.1. *Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets*

Set theory, a cornerstone of mathematics, provides a framework for analyzing collections of elements called "sets" [61,139]. This study examines three major extensions—Fuzzy Sets [288], Neutrosophic Sets [237], and Plithogenic Sets [243]—and their generalizations into Hyperfuzzy [106], HyperNeutrosophic [90], and Hyperplithogenic Sets [90].

These frameworks address various dimensions of uncertainty. *Fuzzy Sets* represent imprecision through membership values between 0 and 1 [288]. *Neutrosophic Sets* enhance this by adding truth, indeterminacy, and falsity components, offering richer analyses of complex systems [237]. *Plithogenic Sets* further extend these ideas to handle multidimensional uncertainty and contradictions, making them particularly effective for analyzing highly complex systems [244, 256].

Takaaki Fujita and Florentin Smarandache, Exploring Concepts of HyperFuzzy, HyperNeutrosophic, and HyperPlithogenic Sets

1.2. *Hyperfuzzy, HyperNeutrosophic, and Hyperplithogenic Sets*

Extensions of Fuzzy Sets [90, 106, 144, 262], Neutrosophic Sets [90], Plithogenic Sets [90], Soft Sets [1, 84, 99, 121, 137, 213, 226, 229, 242, 250], Rough Sets [90], and Vague Sets [90] have been developed using Hyperstructures and n-SuperHyperstructures.

For instance, Fuzzy Sets have been extended into Hyperfuzzy Sets [80, 106, 144–147, 172, 176, 177, 190, 262] and SuperHyperfuzzy Sets [90]. Similarly, Neutrosophic Sets have been extended into HyperNeutrosophic Sets [90, 241] and SuperHyperNeutrosophic Sets [90, 241], while Plithogenic Sets have been extended into HyperPlithogenic Sets [90, 241] and SuperHyperPlithogenic Sets [90, 241].

1.3. *Our Contribution in This Paper*

This section highlights the contributions of this paper. Building on previous research [83], we investigate the potential applications of HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets in various domains.

The study focuses primarily on theoretical exploration and mathematical formulation. For example, we extend concepts such as Neutrosophic Logic, Cognitive Maps, Graph Neural Networks, Classifiers, and Triplet Groups using HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets, and briefly analyze their properties.

Future research should include experimental validation and application-oriented studies to facilitate practical implementation in specific fields. Through this work, we aim to advance this area of study and encourage further exploration and development of related topics.

2. **Preliminaries and Definitions**

This section outlines the essential concepts and definitions necessary for understanding the discussions in this paper. While we aim to present the fundamental ideas concisely, a comprehensive exploration of all related terms lies beyond the scope of this work. Readers are encouraged to consult the cited references for a more in-depth understanding.

2.1. *Basics of Set Theory and Others*

This subsection provides a brief overview of foundational principles in set theory. For a detailed discussion, we recommend standard references such as [123, 139, 143].

Definition 2.1 (Set)**.** [139] A *set* is a well-defined collection of distinct objects, referred to as *elements*. For any object x, it is always determinable whether x is an element of a given set. If x belongs to a set A, this is denoted as $x \in A$. Sets are often represented using curly braces. For example, $A = \{1, 2, 3\}$ represents a set containing the elements 1, 2, and 3.

Definition 2.2 (Subset)**.** [139] A set A is called a *subset* of another set B, written as $A \subseteq B$, if every element of A is also an element of B. This relationship is formally expressed as:

$$A \subseteq B \iff \forall x\,(x \in A \implies x \in B).$$

If $A \subseteq B$ and $A \neq B$, A is referred to as a *proper subset* of B, denoted by $A \subset B$.

Definition 2.3 (Empty Set)**.** [139] The *empty set*, denoted as $\emptyset$, is the unique set containing no elements. It is formally defined as:

$$\forall x\,(x \notin \emptyset).$$

For example, the empty set can be represented as $\emptyset = \{\}$.

Definition 2.4 (Universal Set)**.** [139] The *universal set*, denoted by U, represents the set containing all objects under consideration within a specific context. Any set A under analysis is a subset of U. Formally:

$$A \subseteq U \quad \text{for any set } A.$$

Although some concepts may not have a direct connection to set theory, the following fundamental mathematical definitions will also be employed. As these are basic definitions, readers are encouraged to refer to relevant literature as needed.

Definition 2.5 (Real Numbers)**.** (cf. [70,127]) The set of real numbers, denoted by $\mathbb{R}$, includes all rational and irrational numbers, which can be represented as points on the real number line. Examples are integers, fractions, and roots.

Definition 2.6 (Natural Numbers)**.** (cf. [160]) The set of natural numbers, denoted by $\mathbb{N}$, consists of all positive integers starting from 1:

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Some conventions also include 0, depending on the context.

Definition 2.7 (Homomorphism)**.** (cf. [75,220]) Let $(A, \star)$ and $(B, \circ)$ be two algebraic structures. A *homomorphism* is a function $f : A \to B$ that satisfies:

$$f(a \star a') = f(a) \circ f(a') \quad \text{for all } a, a' \in A.$$

Definition 2.8 (Operation)**.** [139] An *operation* is a function or rule that combines elements of a set S to produce another element within S. Formally, an operation $\circ$ on S is defined as:

$$\circ : S \times S \to S.$$

Examples include addition and multiplication, which are operations on the set of real numbers $\mathbb{R}$.

Definition 2.9 (Binary Operation). [34] A *binary operation* on a set S is a function $* : S \times S \to S$ that combines two elements $a, b \in S$ to produce another element $a * b \in S$. Examples include addition and subtraction, both of which are binary operations on $\mathbb{R}$.

Definition 2.10 (Graph). [64–66] A *graph*, denoted $G = (V, E)$, consists of:

- V: A set of vertices (or nodes).
- E: A set of edges, where each edge is an unordered pair of vertices $\{u, v\}$, $u, v \in V$.

Definition 2.11 (Directed Graph). (cf. [22,66]) A *directed graph*, denoted $G = (V, E)$, consists of:

- V: A set of vertices (or nodes).
- E: A set of directed edges, where each edge is an ordered pair of vertices (u, v), $u, v \in V$.

Definition 2.12 (Matrix). [25] A matrix is a rectangular array of elements arranged in rows and columns, typically denoted as $A = [a_{ij}]$, where a_{ij} represents the element in the i-th row and j-th column.

Definition 2.13 (Adjacency Matrix of a Graph). (cf. [114]) Let $G = (V, E)$ be a graph with vertex set V and edge set E. The adjacency matrix of G, denoted as $A = [a_{ij}]$, is a square matrix of size $|V| \times |V|$, where $|V|$ is the number of vertices in G. Each entry a_{ij} is defined as:

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge from vertex } v_i \text{ to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

For undirected graphs, the adjacency matrix A is symmetric, whereas for directed graphs, A may not be symmetric.

Definition 2.14 (Weight Matrix). (cf. [193, 268]) A weight matrix is a matrix where each element represents a weight or parameter, often used to describe relationships in graphs, neural networks, or optimization problems, such as edge weights or neural connection strengths.

Definition 2.15 (Approximation). (cf. [206]) Approximation refers to the process of representing a function or a value by another function or value that is close to the original within a specified level of accuracy. It is fundamental in numerical analysis and machine learning.

Definition 2.16 (Vector). (cf. [63]) A vector is an ordered tuple of elements, typically from a field $\mathbb{R}$ or $\mathbb{C}$, representing a point in an n-dimensional space or a directed quantity with both magnitude and direction.

This subsection introduces the concepts of Hyperstructure and Superhyperstructure, advanced mathematical frameworks designed to represent hierarchical and multi-layered systems. A *Hyperstructure* builds upon the powerset of a base set to model relationships within collections of elements. Extending this notion, a *Superhyperstructure* utilizes the n-th powerset to represent intricate hierarchical systems across multiple layers [81, 82, 254, 255]. Below, we formalize the n-th powerset and its related constructs.

Definition 2.17 (Base Set)**.** A *base set* is the foundational set S from which powersets and hyperstructures are constructed. Formally:

$$S = \{x \mid x \text{ is an element within the specified domain}\}.$$

All subsets and operations within $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ are derived from the elements of S.

Definition 2.18 (Powerset)**.** [86, 215] The *powerset* of a set S, denoted as $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.19 (n-th Powerset)**.** (cf. [86, 235, 254])

The n-th powerset of a set H, denoted $P_n(H)$, is defined recursively. Starting with the standard powerset, the construction proceeds as:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

The n-th non-empty powerset, denoted $P_n^*(H)$, excludes the empty set:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ is the powerset of H excluding the empty set.

To formalize the concepts of Hyperstructure and Superhyperstructure, we proceed with the following definitions.

Definition 2.20 (Classical Structure)**.** (cf. [235, 254]) A *Classical Structure* is a mathematical framework defined on a non-empty set H equipped with one or more *Classical Operations* that satisfy specific axioms. A *Classical Operation* is a function:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ and H^m represents the m-fold Cartesian product of H. Examples include addition and multiplication in algebraic structures like groups and rings.

Definition 2.21 (Hyperstructure)**.** (cf. [86, 235, 254]) A *Hyperstructure* extends a Classical Structure by operating on the powerset of a base set. Formally:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is its powerset, and $\circ$ is an operation defined on subsets of $\mathcal{P}(S)$.

Definition 2.22 (n-Superhyperstructure)**.** (cf. [235, 254]) An *n-Superhyperstructure* generalizes a Hyperstructure by utilizing the n-th powerset of a base set. It is defined as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ is an operation on elements of $\mathcal{P}_n(S)$.

A representative example of a superhyperstructure is the *SuperHypergraph*, which incorporates advanced elements such as superedges and supervertices, offering a more abstract and versatile framework for hierarchical modeling [38, 85, 90, 91, 93–96, 105, 116, 117, 183, 246, 247, 249, 252, 254]. Additionally, concepts such as *SuperHyperfunction* have also been explored in the literature [251, 253].

2.3. *Fuzzy Set, Hyperfuzzy Set, and Superhyperfuzzy Set*

This subsection presents the formal definitions of Fuzzy Set, Hyperfuzzy Set, and Superhyperfuzzy Set. These concepts extend the traditional notion of fuzzy values into hierarchical structures, offering more refined tools for representing uncertainty.

Definition 2.23 (Fuzzy Set)**.** [288–292] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A *fuzzy relation* on Y is a fuzzy subset δ of $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, δ is called a *fuzzy relation on* τ if:

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 2.24 (Fuzzy Set: Membership in a Fitness Club)**.** Consider the universe $Y = \{\text{John, Alice, Bob, Sarah}\}$, representing a group of people. A fuzzy set τ defines their membership in a fitness club based on their participation level, where:

$$\tau(y) = \begin{cases} 1.0, & \text{if the person is a regular member (e.g., Alice),} \\ 0.8, & \text{if the person participates occasionally (e.g., Bob),} \\ 0.3, & \text{if the person rarely participates (e.g., Sarah),} \\ 0.0, & \text{if the person is not a member (e.g., John).} \end{cases}$$

The mapping $\tau : Y \to [0,1]$ intuitively represents the degree of belonging for each individual in the club.

Definition 2.25 (Hyperfuzzy Set)**.** [29, 106, 144, 189, 262] Let X be a non-empty set. A *hyperfuzzy set* over X is defined as a mapping $\tilde{\mu} : X \to \tilde{P}([0,1])$, where $\tilde{P}([0,1])$ represents the set of all non-empty subsets of the interval $[0,1]$.

Example 2.26 (Hyperfuzzy Set: Customer Satisfaction Ratings)**.** Customer satisfaction ratings are often analyzed from the perspective of fuzzy set theory [76, 76, 135, 166]. Consider a set $X = \{\text{Product A, Product B, Product C}\}$, representing three products. A hyperfuzzy set $\tilde{\mu}$ maps each product to a set of customer satisfaction ratings, where:

$$\tilde{\mu}(x) = \begin{cases} \{0.9, 1.0\}, & \text{for Product A (highly satisfied customers)}, \\ \{0.4, 0.6, 0.8\}, & \text{for Product B (moderately satisfied customers)}, \\ \{0.1, 0.2\}, & \text{for Product C (low satisfaction levels)}. \end{cases}$$

Here, $\tilde{\mu} : X \to \tilde{P}([0,1])$, where each product is associated with a set of satisfaction levels, capturing the diverse opinions of customers.

Definition 2.27 (n-SuperHyperFuzzy Set)**.** [90, 241] Let X be a non-empty set. An n-SuperHyperFuzzy Set is a recursive extension of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets, defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1])$ is similarly defined for the interval $[0,1]$.
- $\tilde{\mu}_n$ maps each element $A \in \tilde{\mathcal{P}}_n(X)$ to a non-empty subset $\tilde{\mu}_n(A) \subseteq [0,1]$, which represents the membership degrees of A at the n-th hierarchical level.

2.4. *Neutrosophic, HyperNeutrosophic, and SuperHyperNeutrosophic Sets*

Neutrosophic Sets enhance Fuzzy Sets by incorporating the concept of indeterminacy, allowing them to model situations that are neither entirely true nor false [237]. This framework offers a more comprehensive approach to handling real-world scenarios characterized by significant uncertainty and complexity, making it a focus of extensive research [100, 101, 155, 236, 238, 240, 248, 257, 258, 260, 261]. The formal definitions are provided below.

Definition 2.28 (Neutrosophic Set)**.** [237] Let X be a non-empty set. A *Neutrosophic Set* A on X is defined by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity for each $x \in X$. These values satisfy the condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 2.29 (Neutrosophic Set: Decision-Making in Hiring)**.** Decision-making is often studied in conjunction with Neutrosophic Sets [6, 11, 52, 60, 164, 181, 198, 286]. Consider $X = \{\text{Candidate A, Candidate B, Candidate C}\}$, representing applicants for a job. A Neutrosophic Set A defines the suitability of each candidate, where:

$$T_A(x), I_A(x), F_A(x)$$

denote the degrees of truth (suitability), indeterminacy (uncertainty), and falsity (unsuitability) for each candidate:

$$T_A(\text{Candidate A}) = 0.8, \quad I_A(\text{Candidate A}) = 0.1, \quad F_A(\text{Candidate A}) = 0.1,$$

$$T_A(\text{Candidate B}) = 0.5, \quad I_A(\text{Candidate B}) = 0.4, \quad F_A(\text{Candidate B}) = 0.1,$$

$$T_A(\text{Candidate C}) = 0.3, \quad I_A(\text{Candidate C}) = 0.2, \quad F_A(\text{Candidate C}) = 0.5.$$

Here, the Neutrosophic Set models the hiring committee's confidence, uncertainty, and rejection levels for each applicant.

Definition 2.30 (HyperNeutrosophic Set)**.** [90] Let X be a non-empty set. A *HyperNeutrosophic Set* on X is a mapping $\tilde{\mu} : X \to \tilde{P}([0,1]^3)$, where $\tilde{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ represents a collection of membership values, with each element comprising degrees of truth (T), indeterminacy (I), and falsity (F). These components satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 2.31 (HyperNeutrosophic Set: Product Feedback Analysis)**.** Consider $X = \{\text{Product X, Product Y, Product Z}\}$, representing three products. A HyperNeutrosophic Set $\tilde{\mu}$ maps each product to a set of customer opinions, where each opinion is a triple (T, I, F) representing truth (positive feedback), indeterminacy (uncertainty), and falsity (negative feedback):

$$\tilde{\mu}(\text{Product X}) = \{(0.9, 0.1, 0.0), (0.8, 0.2, 0.0)\},$$

$$\tilde{\mu}(\text{Product Y}) = \{(0.6, 0.3, 0.1), (0.5, 0.4, 0.1), (0.7, 0.2, 0.1)\},$$

$$\tilde{\mu}(\text{Product Z}) = \{(0.4, 0.5, 0.1), (0.3, 0.6, 0.1)\}.$$

This representation captures the diversity of customer feedback, with multiple sets of opinions reflecting varying perspectives.

Definition 2.32 (n-SuperHyperNeutrosophic Set)**.** [90] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive extension of Neutrosophic and HyperNeutrosophic Sets, defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0,1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,

$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$

  denotes the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is defined analogously for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for A at the n-th hierarchical level.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition holds:

$$0 \leq T + I + F \leq 3.$$

2.5. *HyperPlithogenic Set*

The Plithogenic Set extends traditional set theories, such as Neutrosophic and Fuzzy Sets, by incorporating multi-dimensional attributes and contradictions [243,244]. Below, we present its formal definition.

Definition 2.33 (Plithogenic Set)**.** [243, 244] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF),$$

where:

- v: an attribute.
- Pv: the set of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$: the *Degree of Appurtenance Function (DAF)*, mapping elements and attribute values to a membership degree.
- $pCF : Pv \times Pv \to [0,1]^t$: the *Degree of Contradiction Function (DCF)*, quantifying contradictions between attribute values.

These functions satisfy the following axioms:

(1) *Reflexivity of DCF*:

$$pCF(a, a) = 0, \quad \text{for all } a \in Pv.$$

(2) *Symmetry of DCF*:

$$pCF(a,b) = pCF(b,a), \quad \text{for all } a,b \in Pv.$$

Example 2.34 (Examples of Plithogenic Sets)**.** [87, 98] The Plithogenic Set has various special cases:

- If $s = t = 1$, the set is called a *Plithogenic Fuzzy Set*.
- If $s = 2, t = 1$, it becomes a *Plithogenic Intuitionistic Fuzzy Set*.
- If $s = 3, t = 1$, it is referred to as a *Plithogenic Neutrosophic Set*.

Definition 2.35 (HyperPlithogenic Set)**.** [90] Let X be a non-empty set, and A a set of attributes. For each $v \in A$, let Pv be the range of possible values of v. A *HyperPlithogenic Set HPS* on X is defined as:

$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF),$$

where:

- $P \subseteq X$: a subset of the universe.
- For each $v_i \in A$, Pv_i: the set of possible values for v_i.
- $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$: the *Hyper Degree of Appurtenance Function (HDAF)*, assigning membership degrees as sets.
- $pCF : \bigcup_{i=1}^n Pv_i \times \bigcup_{i=1}^n Pv_i \to [0,1]^t$: the *Degree of Contradiction Function (DCF)*.

Definition 2.36 (n-SuperHyperPlithogenic Set)**.** [90] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes with respective ranges P_{v_i}. An n-SuperHyperPlithogenic Set $SHPS_n$ is defined recursively as:

$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$

where:

- $P_1 \subseteq X$, and for $k \geq 2$,

  $$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$

  representing the k-th nested family of subsets.
- For each v_i, P_{v_i}: the set of possible values of v_i.
- $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$: the *HDAF* at the n-th level.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$: the DCF, satisfying:
  (1) *Reflexivity*: $pCF^{(n)}(a,a) = 0$,
  (2) *Symmetry*: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$.

3. **Result: Application of HyperNeutrosophic Sets to Various Sciences**

In this section, we explore the application of HyperNeutrosophic Sets across various scientific domains, following the approach outlined in [83]. It is important to note that if HyperNeutrosophic Sets prove applicable to a specific domain, it is reasonable to assume that Hyperfuzzy Sets and Hyperplithogenic Sets could also be utilized in similar contexts. Moreover, for SuperHyperNeutrosophic Sets, it is equally logical to investigate the potential applications of SuperHyperfuzzy Sets and SuperHyperplithogenic Sets within the same or related fields.

3.1. *Neutrosophic Logic*

Logic is the systematic study of reasoning, involving principles and rules to distinguish valid arguments, truth, and consistency [54, 72, 233]. Neutrosophic Logic builds upon classical and fuzzy logic [288] by introducing three degrees—truth, indeterminacy, and falsity—allowing for nuanced reasoning under uncertainty [18, 30, 45, 97, 102, 212, 239, 245]. The concept of a Neutrosophic Set can be viewed as an application of Neutrosophic Logic within the framework of set theory. As evident from previous discussions and references such as [83], it is both natural and necessary to explicitly extend Neutrosophic Logic into HyperNeutrosophic Logic and n-SuperHyperNeutrosophic Logic. While the discussion here centers on HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, similar analyses can also be conducted for Hyperfuzzy Sets, n-SuperHyperfuzzy Sets, Hyperplithogenic Sets, and n-SuperHyperplithogenic Sets.

Definition 3.1 (Neutrosophic Logic)**.** [237] Let p be a proposition. In Neutrosophic Logic, the truth value of p is given by an ordered triple

$$v(p) = (T, I, F) \in [0, 1]^3,$$

where T denotes the degree of truth, I denotes the degree of indeterminacy, and F denotes the degree of falsity. These satisfy the following condition:

$$0 \leq T + I + F \leq 3.$$

Unlike many-valued logics that fix $T + F \leq 1$, Neutrosophic Logic allows T, I, F to vary somewhat independently, thereby capturing paradoxical and uncertain statements more flexibly.

Example 3.2 (Neutrosophic Example)**.** Consider a proposition p with

$$v(p) = (0.7,\ 0.2,\ 0.4).$$

Here $T = 0.7, I = 0.2, F = 0.4$, and $0.7 + 0.2 + 0.4 = 1.3 \leq 3$. Thus p can be viewed as mostly true, with moderate falsity and some degree of indeterminacy.

Definition 3.3 (HyperNeutrosophic Logic)**.** Let p be a proposition. In HyperNeutrosophic Logic, the truth value of p is given by a non-empty subset of $[0,1]^3$:

$$v(p) \subseteq [0,1]^3, \quad v(p) \neq \varnothing,$$

where each element $(T, I, F) \in v(p)$ satisfies $0 \leq T + I + F \leq 3$.

Example 3.4 (HyperNeutrosophic Example)**.** Suppose we have two expert opinions about p. One expert assigns $(T, I, F) = (0.7, 0.2, 0.4)$, and another expert assigns $(T, I, F) = (0.4, 0.1, 0.8)$. Then the HyperNeutrosophic valuation can be taken as

$$v(p) = \{(0.7,\, 0.2,\, 0.4),\ (0.4,\, 0.1,\, 0.8)\}.$$

This set-based valuation captures multiple sources of uncertain or even conflicting information.

Definition 3.5 (n-SuperHyperNeutrosophic Logic)**.** Let X be a non-empty set and define recursively

$$\tilde{\mathcal{P}}_1(X) = \{A \subseteq X : A \neq \varnothing\}, \quad \tilde{\mathcal{P}}_k(X) = \{B \subseteq \tilde{\mathcal{P}}_{k-1}(X) : B \neq \varnothing\} \quad (k \geq 2).$$

A *n-SuperHyperNeutrosophic valuation* $v(p)$ is defined to be an element of

$$\tilde{\mathcal{P}}_n([0,1]^3),$$

i.e. an n-th level nested non-empty subset of the unit cube $[0,1]^3$. At every level, each $(T, I, F) \in [0,1]^3$ must satisfy $0 \leq T + I + F \leq 3$.

Example 3.6 ($n = 2$ SuperHyperNeutrosophic Example)**.** An example of a 2-SuperHyperNeutrosophic valuation $v(p)$ could be

$$v(p) = \Big\{\{(0.7,\, 0.2,\, 0.4), (0.4,\, 0.1,\, 0.8)\},\ \{(0.6,\, 0.3,\, 0.2)\}\Big\}.$$

Here each inner set, such as $\{(0.7, 0.2, 0.4), (0.4, 0.1, 0.8)\}$, is itself a valid HyperNeutrosophic subset of $[0,1]^3$. We then collect those subsets into a larger non-empty set, forming a second-level structure.

Theorem 3.7. *It holds as follows.*

(1) When $n = 1$, an n-SuperHyperNeutrosophic valuation is exactly a HyperNeutrosophic valuation.

(2) If we restrict a HyperNeutrosophic valuation to be a singleton $\{(T, I, F)\}$, it recovers Neutrosophic Logic.

Hence n-SuperHyperNeutrosophic Logic generalizes both HyperNeutrosophic and Neutrosophic Logics.

Proof. *(1)* By definition, for $n = 1$ we have

$$\tilde{\mathcal{P}}_1([0,1]^3) = \{ A \subseteq [0,1]^3 : A \neq \varnothing \}.$$

Thus a 1-SuperHyperNeutrosophic valuation $v(p)$ is simply a non-empty subset of $[0,1]^3$, which is precisely the definition of a HyperNeutrosophic valuation.

(2) In Neutrosophic Logic, $v(p)$ is a single triple $(T, I, F) \in [0,1]^3$. If we embed it into HyperNeutrosophic Logic by forming the singleton $\{(T, I, F)\}$, this is clearly a non-empty subset of $[0,1]^3$, thus satisfying the HyperNeutrosophic requirements. Therefore, the singleton case of HyperNeutrosophic valuations coincides with Neutrosophic valuations.

Combining these, we see that n-SuperHyperNeutrosophic Logic (for $n = 1$) equals HyperNeutrosophic Logic, while Neutrosophic Logic is recovered as the singleton case within HyperNeutrosophic sets. For $n > 1$, the framework further generalizes these logics by allowing nested families of HyperNeutrosophic sets. □

3.2. *HyperNeutrosophic Graph Neural Network*

A neural network is a computational model inspired by biological neural systems, designed for tasks such as pattern recognition, data classification, and prediction [8,13,23,159,274,282,283]. Building upon this foundation, a Graph Neural Network (GNN) extends neural networks to graph structures, enabling the modeling of relationships between nodes, edges, and their associated features [58, 142, 187, 211, 227, 231, 272, 278, 295, 299]. Readers may refer to the lecture notes or the introduction for further details(cf. [2,58,74,142,187,211,227,231,284,295]). Building on this concept, Hypergraph Neural Networks (HGNNs) extend traditional Graph Neural Networks (GNNs) by utilizing hyperedges to model higher-order relationships involving multiple nodes simultaneously [35, 79, 122, 128, 141, 267, 276]. Additionally, related concepts, such as the n-SuperHypergraph Neural Network, have also been proposed [86].

Considering these aspects, this paper examines Hyperneutrosophic Graph Neural Networks and Superhyperneutrosophic Graph Neural Networks (cf. [86]). First, several graph concepts addressing various types of uncertainty are briefly introduced below.

Definition 3.8 (Unified Framework for Uncertain Graphs)**.** (cf. [88]) Let $G = (V, E)$ be a classical graph, where V is the set of vertices and E is the set of edges. Depending on the type of graph, each vertex $v \in V$ and edge $e \in E$ is associated with membership values to represent various degrees of truth, indeterminacy, falsity, and other measures of uncertainty.

(1) *Fuzzy Graph* (cf. [26, 103, 107, 186, 196, 217, 278])
   - Each vertex $v \in V$ is assigned a membership degree $\sigma(v) \in [0,1]$.
   - Each edge $e = (u, v) \in E$ is assigned a membership degree $\mu(u, v) \in [0,1]$.

(2) *Intuitionistic Fuzzy Graph (IFG)* (cf. [7, 138, 270, 297])

- Each vertex $v \in V$ has two values: $\mu_A(v) \in [0, 1]$ (degree of membership) and $\nu_A(v) \in [0, 1]$ (degree of non-membership), satisfying $\mu_A(v) + \nu_A(v) \leq 1$.
- Each edge $e = (u, v) \in E$ has two values: $\mu_B(u, v) \in [0, 1]$ and $\nu_B(u, v) \in [0, 1]$, with $\mu_B(u, v) + \nu_B(u, v) \leq 1$.

(3) *Neutrosophic Graph* (cf. [32, 33, 113, 129, 150, 247, 259])

- Each vertex $v \in V$ is associated with a triplet

$$\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$$

, where

$$\sigma_T(v), \sigma_I(v), \sigma_F(v) \in [0, 1]$$

and $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 3$.

- Each edge $e = (u, v) \in E$ is associated with a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$.

(4) *Quadripartitioned Neutrosophic Graph (QNG)* (cf. [131–133, 225, 232])

- Each vertex $v \in V$ is associated with a quadripartitioned neutrosophic membership

$$\sigma(v) = (\sigma_1(v), \sigma_2(v), \sigma_3(v), \sigma_4(v))$$

, where

$$\sigma_1(v), \sigma_2(v), \sigma_3(v), \sigma_4(v) \in [0, 1]$$

and

$$\sigma_1(v) + \sigma_2(v) + \sigma_3(v) + \sigma_4(v) \leq 4$$

.

- Each edge $e = (u, v) \in E$ is associated with a quadripartitioned membership

$$\sigma(e) = (\sigma_1(e), \sigma_2(e), \sigma_3(e), \sigma_4(e))$$

, satisfying:

$$\sigma_1(e) \leq \min\{\sigma_1(u), \sigma_1(v)\},$$

$$\sigma_2(e) \leq \min\{\sigma_2(u), \sigma_2(v)\},$$

$$\sigma_3(e) \leq \max\{\sigma_3(u), \sigma_3(v)\},$$

$$\sigma_4(e) \leq \max\{\sigma_4(u), \sigma_4(v)\}.$$

Example 3.9 (Fuzzy Graph)**.** Let $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_2), (v_2, v_3)\}$. Each vertex $v \in V$ is assigned a membership degree:

$$\sigma(v_1) = 0.8, \quad \sigma(v_2) = 0.5, \quad \sigma(v_3) = 0.7.$$

Each edge $e \in E$ is assigned a membership degree:

$$\mu(v_1, v_2) = 0.6, \quad \mu(v_2, v_3) = 0.9.$$

This defines a Fuzzy Graph where vertices and edges have varying degrees of membership.

Example 3.10 (Neutrosophic Graph)**.** Let $G = (V, E)$, where $V = \{v_1, v_2, v_3\}$ and $E = \{(v_1, v_2), (v_2, v_3)\}$. Each vertex $v \in V$ is associated with a triplet $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$:

$$\sigma(v_1) = (0.7, 0.2, 0.1), \quad \sigma(v_2) = (0.6, 0.3, 0.1), \quad \sigma(v_3) = (0.8, 0.1, 0.1).$$

Each edge $e \in E$ is associated with a triplet $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$:

$$\mu(v_1, v_2) = (0.5, 0.3, 0.2), \quad \mu(v_2, v_3) = (0.6, 0.2, 0.2).$$

This defines a Neutrosophic Graph with truth, indeterminacy, and falsity values for vertices and edges.

The Neutrosophic Graph Neural Network, along with its extensions, the HyperNeutrosophic Graph Neural Network and the SuperHyperNeutrosophic Graph Neural Network, are introduced below. While the discussion here centers on HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, similar analyses can also be conducted for Hyperfuzzy Sets, n-SuperHyperfuzzy Sets, Hyperplithogenic Sets, and n-SuperHyperplithogenic Sets.

Definition 3.11. In general, feature spaces represent the domains of attributes for vertices and edges, denoted by X_V and X_E, respectively [21, 56, 151]. Aggregation rules are operations that combine features or information from multiple elements, such as vertices or edges, into a unified representation, denoted as $\mathcal{R}_N$ (cf. [43, 167, 188]). Learnable parameters, denoted as Θ, are adjustable variables (e.g., weights in neural networks) optimized during training to improve model performance (cf. [300]).

Definition 3.12 (Neutrosophic Graph Neural Network (N-GNN) [88])**.** A *Neutrosophic Graph Neural Network (N-GNN)* is a GNN that leverages neutrosophic logic to handle uncertain, indeterminate, and inconsistent data in graph-structured settings. Formally, an N-GNN is an 8-tuple:

$$\text{N-GNN} = \Big(G, X_V, X_E, \mathcal{N}_V, \mathcal{N}_E, \mathcal{R}_N, \mathcal{D}_N, \Theta\Big),$$

where:

(1) $G = (V, E)$ is a graph with vertex set V and edge set E.
(2) X_V and X_E are the feature spaces for vertices and edges, respectively.
(3) $\mathcal{N}_V : X_V \to [0, 1]^3$ and $\mathcal{N}_E : X_E \to [0, 1]^3$ are *neutrosophic fuzzification functions*, mapping features to triples (T, I, F) satisfying $T + I + F \leq 3$.
(4) $\mathcal{R}_N$ is a set of *neutrosophic aggregation rules* specifying how neutrosophic information is combined among vertices and edges.

(5) $\mathcal{D}_N$ is a *neutrosophic defuzzification function* that transforms aggregated neutrosophic values into a crisp or probabilistic output (e.g., a real number or a probability vector [174]).

(6) Θ is the set of learnable parameters (e.g., weights in neural layers or rule parameters).

N-GNN Layer. Given a vertex feature $x_v \in X_V$ for $v \in V$ and an edge feature $x_{uv} \in X_E$ for an edge $(u, v) \in E$, the neutrosophic fuzzification layer outputs:

$$\mathcal{N}_V(x_v) = (\mu_T^v, \mu_I^v, \mu_F^v), \quad \mathcal{N}_E(x_{uv}) = (\mu_T^{uv}, \mu_I^{uv}, \mu_F^{uv}),$$

where each triple fulfills $\mu_T + \mu_I + \mu_F \leq 3$.

Neutrosophic Aggregation. Let $\mathrm{AGG}_N(\cdot)$ be a neutrosophic aggregation operator guided by the rule set $\mathcal{R}_N$. For a vertex v, a typical update rule might be:

$$h_v^{(l+1)} = \sigma\Big(\mathrm{AGG}_N\big(\{ h_u^{(l)}, \mathcal{N}_E(x_{uv}) \mid u \in \mathcal{N}(v)\}\big)\Big),$$

where $h_v^{(l)}$ denotes the hidden representation of vertex v at layer l, $\mathcal{N}(v)$ is the neighborhood of v, and σ is a non-linear activation function [68, 230] (e.g., ReLU). After several layers, the defuzzification step $\mathcal{D}_N$ produces a final crisp or probabilistic output.

Example 3.13 (A Simple N-GNN on a Triangular Graph)**.** *Scenario:* Suppose we have a small graph $G = (V, E)$ with three vertices $V = \{A, B, C\}$ and three edges $E = \{(A, B), (B, C), (C, A)\}$ (cf. [264]). Each vertex and edge has certain uncertain features that we wish to model using neutrosophic logic.

(1) *Vertex Features:*

Let us assume each vertex v has a single feature x_v (e.g., an uncertain sensor reading [77]). We define:

$$x_A = 0.7, \quad x_B = 0.5, \quad x_C = 0.9.$$

Since these sensor readings contain some noise or uncertainty, we convert them into neutrosophic triples (T, I, F) as follows:

$$\mathcal{N}_V(x_A) = (0.6,\ 0.3,\ 0.1), \quad \mathcal{N}_V(x_B) = (0.4,\ 0.2,\ 0.4), \quad \mathcal{N}_V(x_C) = (0.8,\ 0.1,\ 0.1).$$

Each triple must satisfy $T + I + F \leq 3$; here they all sum to $1.0 \leq 3$.

(2) *Edge Features:*

Each edge $(u, v) \in E$ also has a feature x_{uv} (e.g., an uncertain measure of connection strength). For simplicity:

$$x_{AB} = 0.2, \quad x_{BC} = 0.6, \quad x_{CA} = 0.4.$$

Using the neutrosophic fuzzification $\mathcal{N}_E$, suppose:

$$\mathcal{N}_E(x_{AB}) = (0.3,\ 0.4,\ 0.3), \quad \mathcal{N}_E(x_{BC}) = (0.5,\ 0.3,\ 0.2), \quad \mathcal{N}_E(x_{CA}) = (0.4,\ 0.2,\ 0.4).$$

(3) *Initial Hidden States:*

Assign an initial hidden representation $h_v^{(0)} \in \mathbb{R}^d$ to each vertex (e.g., $d = 2$ dimensions). For instance:

$$h_A^{(0)} = (1.0,\, 0.0), \quad h_B^{(0)} = (0.5,\, 0.5), \quad h_C^{(0)} = (0.0,\, 1.0).$$

(4) *Neutrosophic Aggregation:*

We define a neutrosophic aggregation rule AGG_N that combines:

$$(\mu_T^v, \mu_I^v, \mu_F^v) \quad \text{for } v \in V \quad \text{and} \quad (\mu_T^{uv}, \mu_I^{uv}, \mu_F^{uv}) \quad \text{for } (u, v) \in E$$

using some operator, e.g., neutrosophic min or a product-based approach adapted for (T, I, F).

A simplified update for vertex A at layer 1 might look like:

$$h_A^{(1)} = \sigma\Big(W \cdot \mathrm{AGG}_N\big(\{\, h_B^{(0)}, \mathcal{N}_E(x_{AB}),\ h_C^{(0)}, \mathcal{N}_E(x_{CA})\}\big)\Big),$$

where W is a weight matrix, and σ is an activation function (like ReLU).

(5) *Final Defuzzification:*

After 2 or 3 message-passing layers, each $h_v^{(\text{final})}$ might be defuzzified via $\mathcal{D}_N$ to produce a class label (e.g., in a classification setting) or a real-valued score (in a regression setting). For example, one could aggregate the final (T, I, F) into a single confidence measure by $\mu_T - \mu_F$ or other transformations, and then map $h_v^{(\text{final})}$ to a label.

(6) *Interpretation:*

A higher T in (T, I, F) suggests the data is more likely to be "true" or valid. A higher I indicates indeterminacy or lack of clarity. A higher F signals contradictions or false components.

By tracking these three degrees, the N-GNN can learn to handle nodes or edges with uncertain or conflicting information more effectively than a standard GNN.

This small triangular graph example shows how even a beginner can view Neutrosophic Graph Neural Networks in action. Each vertex and edge has a neutrosophic triple representing its uncertain state, and the GNN aggregates these values through specialized neutrosophic rules. The final output offers a robust way to manage uncertainty, indeterminacy, and contradiction in the data.

Definition 3.14 (HyperNeutrosophic Graph Neural Network (HN-GNN))**.** A *HyperNeutrosophic Graph Neural Network (HN-GNN)* generalizes the N-GNN by allowing each vertex or edge to have a *set* of neutrosophic triples, rather than a single triple. Formally, an HN-GNN is a 9-tuple:

$$\text{HN-GNN} \;=\; \Big(G,\ X_V,\ X_E,\ \mathcal{HN}_V,\ \mathcal{HN}_E,\ \mathcal{R}_{HN},\ \mathcal{D}_{HN},\ \mathrm{AGG}_{set},\ \Theta\Big),$$

where:

(1) $G = (V, E)$ is a graph.
(2) X_V, X_E are vertex and edge feature spaces.
(3) $\mathcal{HN}_V : X_V \to \mathcal{P}([0,1]^3)$ and $\mathcal{HN}_E : X_E \to \mathcal{P}([0,1]^3)$ are *hyperneutrosophic fuzzification functions*, mapping each vertex (or edge) to a non-empty subset of $[0,1]^3$. Each element $(T_k, I_k, F_k) \in \mathcal{HN}_V(x_v)$ (or $\mathcal{HN}_E(x_{uv})$) satisfies $T_k + I_k + F_k \leq 3$.
(4) AGG_{set} is a *set-level aggregation operator* that collapses or summarizes each hyperneutrosophic set into either (a) a single representative triple, or (b) a small set of representative triples used in the subsequent GNN computation.
(5) $\mathcal{R}_{HN}$ is the hyperneutrosophic rule set for combining hyperneutrosophic information from neighboring vertices and edges.
(6) $\mathcal{D}_{HN}$ is the hyperneutrosophic defuzzification function, producing a final crisp output from the hyperneutrosophic representations.
(7) Θ is the set of learnable parameters in the model.

HN-GNN Layer. At each layer l, for a vertex v:

$$\mathcal{HN}_V(x_v) \;=\; \Big\{(\mu_T^v(i),\, \mu_I^v(i),\, \mu_F^v(i)) \;\Big|\; i \in \mathcal{I}_v\Big\} \;\subseteq\; [0,1]^3,$$

where $\mathcal{I}_v$ indexes multiple neutrosophic evaluations. An edge (u, v) has:

$$\mathcal{HN}_E(x_{uv}) \;=\; \Big\{(\mu_T^{uv}(j),\, \mu_I^{uv}(j),\, \mu_F^{uv}(j)) \;\Big|\; j \in \mathcal{J}_{uv}\Big\}.$$

We first aggregate each hyperneutrosophic set into a suitable representation (e.g., average or maximum triple), or keep multiple triples for a richer representation. The node update then proceeds similarly to an N-GNN, but with set-based inputs instead of single triples:

$$h_v^{(l+1)} \;=\; \sigma\Big(\mathrm{AGG}_{HN}\big(\{\, h_u^{(l)}, \mathcal{HN}_E(x_{uv}) \mid u \in \mathcal{N}(v)\}\big)\Big).$$

Definition 3.15 (n-SuperHyperNeutrosophic Graph Neural Network (n-SHN-GNN))**.** An *n-SuperHyperNeutrosophic Graph Neural Network (n-SHN-GNN)* is a further generalization of the HN-GNN, in which each vertex or edge is endowed with an *n-SuperHyperNeutrosophic Set* instead of a HyperNeutrosophic Set. Formally, an n-SHN-GNN is a 9-tuple:

$$\text{n-SHN-GNN} \;=\; \Big(G,\; X_V,\; X_E,\; \mathcal{SHN}_V^{(n)},\; \mathcal{SHN}_E^{(n)},\; \mathcal{R}_{SHN}^{(n)},\; \mathcal{D}_{SHN}^{(n)},\; \mathrm{AGG}_n,\; \Theta\Big),$$

where:

(1) $G = (V, E)$ is a graph.
(2) X_V, X_E are vertex and edge feature spaces.
(3) $\mathcal{SHN}_V^{(n)} : X_V \to \tilde{\mathcal{P}}_n([0,1]^3)$ and $\mathcal{SHN}_E^{(n)} : X_E \to \tilde{\mathcal{P}}_n([0,1]^3)$ map each vertex (or edge) to an n-SuperHyperNeutrosophic Set of neutrosophic triples. Concretely, each vertex

(or edge) is associated with an n-th nested family of subsets of $[0,1]^3$. Each triple (T, I, F) must satisfy $T + I + F \leq 3$.

(4) AGG_n is an aggregation operator that *collapses* each n-SuperHyperNeutrosophic Set into a small number of representative neutrosophic triples (e.g., using hierarchical combination rules).

(5) $\mathcal{R}_{SHN}^{(n)}$ is the rule set for combining n-SuperHyperNeutrosophic information among neighbors.

(6) $\mathcal{D}_{SHN}^{(n)}$ is the defuzzification step, producing the final crisp or fuzzy outputs from the aggregated hierarchical sets.

(7) Θ is the set of trainable parameters.

Hierarchical Set Representation. Let $\tilde{\mathcal{P}}_n([0,1]^3)$ denote the n-th nested power set of the neutrosophic cube. For a vertex v:

$$\mathcal{SHN}_V^{(n)}(x_v) \;\in\; \tilde{\mathcal{P}}_n([0,1]^3),$$

which might be represented recursively:

$$\mathcal{SHN}_V^{(1)}(x_v) \;=\; \tilde{\mathcal{P}}\big([0,1]^3\big), \quad \mathcal{SHN}_V^{(2)}(x_v) \;=\; \tilde{\mathcal{P}}\Big(\mathcal{SHN}_V^{(1)}(x_v)\Big), \quad \ldots$$

At each level, sets of sets of neutrosophic triples are nested, capturing multi-level uncertainties or multi-source conflicting information.

Layer Update in an n-SHN-GNN. At layer l, suppose each vertex v has hidden representation $h_v^{(l)}$. To update $h_v^{(l+1)}$, do:

$$h_v^{(l+1)} \;=\; \sigma\Big(\mathrm{AGG}_n\Big(\big\{\mathcal{SHN}_E^{(n)}(x_{uv}),\, h_u^{(l)}\big\}_{u \in \mathcal{N}(v)}\Big)\Big).$$

Here, AGG_n must systematically process the nested hierarchical sets from each edge (u, v) or from the vertex features $\mathcal{SHN}_V^{(n)}(x_v)$. After a user-defined number of layers, $\mathcal{D}_{SHN}^{(n)}$ is applied to produce the final output (e.g., classification scores or regression values).

Key Properties of an n-SHN-GNN:

- *Deep Hierarchical Uncertainty:* The n-th nested sets encode multiple layers of contradictory, uncertain, or aggregated data sources.
- *Flexible Aggregation:* Each level requires a well-defined rule to merge or reduce the hierarchical sets into workable forms for neural computations.
- *Generalization of All Previous Cases:* Setting $n = 0$ or $n = 1$ reduces to classical or hyperneutrosophic graph neural networks, respectively, thus unifying these frameworks under one hierarchy.

Remark 3.16. The above definitions of N-GNN, HN-GNN, and n-SHN-GNN assume typical forward-pass, layer-by-layer neural network operations. Training is done by gradient-based optimization (e.g., backpropagation [120,170,216,279]) on a loss function that measures predictive

performance. The novel aspect is the representation of edges and vertices with (hyper)neutrosophic or n-superhyperneutrosophic sets of membership values, enabling richer modeling of uncertainty and ambiguity in graph-structured data.

Theorem 3.17 (Generalization Property)**.** *An n-SuperHyperNeutrosophic Graph Neural Network (n-SHN-GNN) strictly generalizes both the HyperNeutrosophic Graph Neural Network (HN-GNN) and the Neutrosophic Graph Neural Network (N-GNN). Specifically:*

- *If $n = 1$, the n-SHN-GNN reduces to the HN-GNN.*
- *If $n = 0$, the n-SHN-GNN reduces to the N-GNN.*

Proof. Case $n = 0$: By definition, an n-SuperHyperNeutrosophic Set becomes a single neutrosophic triple $(T, I, F) \in [0, 1]^3$ if $n = 0$. Consequently, every vertex or edge in the (0)-SHN-GNN is associated with a single neutrosophic triple, which matches exactly the data representation in a standard Neutrosophic Graph Neural Network (N-GNN). Hence, a (0)-SHN-GNN is identical to an N-GNN in all respects (membership representation, aggregator design, defuzzification steps, etc.).

Case $n = 1$: If $n = 1$, the membership for each vertex or edge is a nonempty subset of $[0, 1]^3$, i.e. a HyperNeutrosophic Set, rather than an n-th nested structure. Thus, the architecture becomes exactly that of a HyperNeutrosophic Graph Neural Network (HN-GNN), where each vertex/edge can hold multiple neutrosophic triples simultaneously but not nested sets-of-sets. Hence, a (1)-SHN-GNN is isomorphic to an HN-GNN.

Case $n > 1$: In this situation, each vertex or edge is assigned an n-fold nested hyperstructure of neutrosophic triples, providing a strictly richer representation than either HN-GNN ($n = 1$) or N-GNN ($n = 0$). Therefore, n-SHN-GNN ($n > 1$) strictly generalizes both HN-GNN and N-GNN, as it can simulate them by appropriate "flattening" of membership sets or by choosing $n = 0, 1$. □

Notation 1. *For brevity, let*

$$\text{n-SHN-GNN} = \big(G,\ X_V,\ X_E,\ \mathcal{SHN}_V^{(n)},\ \mathcal{SHN}_E^{(n)},\ \mathcal{R}_{SHN}^{(n)},\ \mathcal{D}_{SHN}^{(n)},\ \mathrm{AGG}_n,\ \Theta\big)$$

be our canonical reference model.

Theorem 3.18 (Well-Definedness of Layer Updates)**.** *Let* AGG_n *be an aggregation operator that maps from*

$$\Big(\tilde{\mathcal{P}}_n\big([0, 1]^3\big)\Big)^{|\mathcal{N}(v)|+1} \to \tilde{\mathcal{P}}_n\big([0, 1]^3\big) \quad \text{or} \quad [0, 1]^d,$$

for some $d \in \mathbb{N}$. Suppose AGG_n *is* closed *under the domain of membership sets and preserves the condition $T + I + F \leq 3$. Then each layer update in an n-SHN-GNN is well-defined:*

$$h_v^{(l+1)} = \sigma\Big(\mathrm{AGG}_n\big(\{\mathcal{SHN}_E^{(n)}(x_{uv}),\ h_u^{(l)}\}_{u \in \mathcal{N}(v)}\big)\Big),$$

yields valid $h_v^{(l+1)}$ in the intended codomain (e.g., $[0,1]^d$).

Proof. By assumption, AGG_n takes as input a finite set of objects that are each either:

- Elements of $\tilde{\mathcal{P}}_n([0,1]^3)$, i.e. n-SuperHyperNeutrosophic sets.
- Real vector embeddings $h_u^{(l)}$ from the preceding layer (if the aggregator merges representation vectors directly).

Since AGG_n is assumed to be closed under the domain of membership sets, it produces an output that remains in $\tilde{\mathcal{P}}_n([0,1]^3)$ (or in $[0,1]^d$). Furthermore, each triple (T, I, F) within the aggregator's output is guaranteed to satisfy $T + I + F \leq 3$. Thus, the output is a *well-defined* n-superhyperneutrosophic representation or a standard vector embedding, suitable for subsequent neural network operations or final defuzzification. The activation function σ (e.g. ReLU) preserves the property of valid real vector outputs or set membership constraints, concluding the well-definedness of each layer update. □

Theorem 3.19 (Continuity of the Forward Pass)**.** *Assume each aggregator* AGG_n *and activation function σ in the n-SHN-GNN is continuous. Then, as a function of the input features* $\{x_v\}_{v\in V}$ *and* $\{x_{uv}\}_{(u,v)\in E}$*, the final output of the n-SHN-GNN is continuous.*

Proof. Let L denote the number of layers, and write

$$\{h_v^{(l)}\}_{v\in V} \quad \text{for} \quad l = 0, 1, \ldots, L.$$

At $l = 0$, we have $h_v^{(0)} = \mathrm{Enc}\big(\mathcal{SHN}_V^{(n)}(x_v)\big)$ or a direct embedding of the vertex features, which is continuous by assumption of the encoding function Enc. The aggregator AGG_n is continuous in its arguments, and σ is also continuous. Therefore, each update

$$h_v^{(l+1)} = \sigma\Big(\mathrm{AGG}_n(\{h_u^{(l)}, \mathcal{SHN}_E^{(n)}(x_{uv})\}_{u\in\mathcal{N}(v)})\Big)$$

is a composition of continuous mappings in terms of $\{h_u^{(l)}\}$ and the input sets $\{\mathcal{SHN}_E^{(n)}(x_{uv})\}$. By induction on the layer index l, continuity is preserved at each layer, culminating in a continuous final output $\{h_v^{(L)}\}_{v\in V}$. Hence the entire forward pass from input feature sets $\{x_v, x_{uv}\}$ to the final output $\{h_v^{(L)}\}$ is continuous. □

Theorem 3.20 (Reduction Homomorphism for Layer Mapping)**.** *Let $\rho_{n\to m}$ be a map $\tilde{\mathcal{P}}_n([0,1]^3) \to \tilde{\mathcal{P}}_m([0,1]^3)$ with $m < n$, defined by recursively selecting or aggregating subsets in the nested structure. Suppose each layer aggregator* AGG_n *commutes with $\rho_{n\to m}$. Then the n-SHN-GNN naturally reduces to an m-SHN-GNN.*

Proof. Define the reduction map $\rho_{n\to m}$ such that for each $A \in \tilde{\mathcal{P}}_n([0,1]^3)$, we find a corresponding $B \in \tilde{\mathcal{P}}_m([0,1]^3)$. Concretely, $\rho_{n\to m}$ flattens the nested subsets from level n down to level m by either discarding certain nesting levels or merging them. If each layer aggregator AGG_n satisfies

$$\rho_{n\to m}\Big(\mathrm{AGG}_n(\{A_i\}_{i\in I})\Big) = \mathrm{AGG}_m(\{\rho_{n\to m}(A_i)\}_{i\in I}),$$

then we have commutativity of aggregator and flattening. Hence, after applying $\rho_{n\to m}$ to each vertex/edge membership set at every layer, the system evolves exactly as if it were an m-SHN-GNN. Consequently, the entire forward pass of the n-SHN-GNN, under $\rho_{n\to m}$, produces the same outputs as the m-SHN-GNN using the aggregator AGG_m. This proves that n-SHN-GNN reduces to an m-SHN-GNN under the existence of such a homomorphism $\rho_{n\to m}$. □

Definition 3.21 (Fixed Point)**.** (cf. [111]) A fixed point of a function $f : X \to X$ is an element $x^* \in X$ such that $f(x^*) = x^*$. Fixed points represent states or solutions where the application of the function leaves the element unchanged.

Theorem 3.22 (Existence and Uniqueness of a Fixed Point under Contractive Aggregation)**.** *Assume each vertex update in an n-SHN-GNN is given by a contraction mapping in the space of real embeddings (or suitably metricized set space). Formally, suppose there exists $\lambda \in (0,1)$ such that for all pairs of states $\mathbf{H}, \mathbf{H}' \in \mathcal{X}^{|V|}$,*

$$d\Big(\mathrm{AGG}_n(\mathbf{H}),\ \mathrm{AGG}_n(\mathbf{H}')\Big)\ \le\ \lambda\, d(\mathbf{H}, \mathbf{H}'),$$

where d is a metric on the space of states. Then there exists a unique fixed point $\mathbf{H}^*$ such that

$$\mathbf{H}^* = \mathrm{AGG}_n\big(\mathbf{H}^*\big).$$

Proof. This theorem is a direct application of the Banach Fixed Point Theorem [110, 126, 192] (or Contraction Mapping Principle [31]). The aggregator AGG_n is interpreted as a function on the entire set of node states $\mathbf{H} \in \mathcal{X}^{|V|}$. By assumption, it is a λ-contraction with $\lambda < 1$. Therefore, there exists a unique fixed point $\mathbf{H}^*$ satisfying $\mathbf{H}^* = \mathrm{AGG}_n(\mathbf{H}^*)$. Existence follows from standard contraction mapping arguments, and uniqueness arises because any other fixed point would produce a contradiction to the strict contraction property. □

Definition 3.23 (Universal Approximation)**.** (cf. [118, 202]) The Universal Approximation Theorem states that a sufficiently large neural network with appropriate activation functions can approximate any continuous function to arbitrary accuracy on a compact domain. This property underlies the expressive power of neural networks in learning complex mappings.

Theorem 3.24 (Universal Approximation of n-Nested Uncertainty)**.** *Let $\mathcal{F}$ be a class of target functions that map from $\tilde{\mathcal{P}}_n([0,1]^3)$-structured input to real output, i.e.,*

$$f : \tilde{\mathcal{P}}_n([0,1]^3) \times \cdots \times \tilde{\mathcal{P}}_n([0,1]^3) \to \mathbb{R}.$$

Suppose each aggregator AGG_n can be realized as a universal approximator for functions over $\tilde{\mathcal{P}}_n([0,1]^3)$. Then an n-SHN-GNN with sufficient hidden layer width and depth can approximate any target function $f \in \mathcal{F}$ arbitrarily well.

Proof. This statement extends the universal approximation property of neural networks to the domain of nested uncertain sets $\tilde{\mathcal{P}}_n([0,1]^3)$. The key idea is that the aggregator AGG_n (plus any standard feedforward sub-layers) must have enough expressive capacity to approximate arbitrary continuous mappings of the inputs from $\tilde{\mathcal{P}}_n([0,1]^3)$. Under standard assumptions of neural universal approximation (e.g., multi-layer perceptrons with sufficient width and suitable activation), we can embed or encode each nested set structure into a finite-dimensional space, apply a universal approximator, and decode as necessary. Provided the aggregator supports transformations rich enough (e.g., a deep parametric function), it can approximate any continuous function on the domain $\tilde{\mathcal{P}}_n([0,1]^3)$. This argument follows the usual universal approximation theorem, adapted to an embedding space for the nested sets. Convergence in approximation is then guaranteed by classical results on feedforward networks with continuous activation functions (e.g., sigmoids or ReLU). □

3.3. *Neutrosophic Cognitive Maps*

A Cognitive Map is a directed graph that models concepts (nodes) and their causal relationships (edges), where the edges are assigned weighted influences [19, 203, 223, 280]. Over time, various extensions have been developed, including Fuzzy Cognitive Maps [17, 162, 168, 199, 201, 210], Intuitionistic Fuzzy Cognitive Maps [69, 134, 175, 200], Neutrosophic Cognitive Maps [9, 148, 185, 194, 207], Dynamic Cognitive Maps [37, 180], Hesitant fuzzy Cognitive Maps [49–51], Rough Cognitive Maps [46, 47], Cognitive Hypermaps [92], and Cognitive n-SuperHypermaps [92].

This subsection focuses on the HyperNeutrosophic Cognitive Map and the n-SuperHyperNeutrosophic Cognitive Map. Their definitions, associated theorems, and relevant properties are detailed below. While the discussion here centers on HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, similar analyses can also be conducted for Hyperfuzzy Sets, n-SuperHyperfuzzy Sets, Hyperplithogenic Sets, and n-SuperHyperplithogenic Sets.

Definition 3.25 (Limit Cycle)**.** (cf. [14]) In general, a limit cycle is a closed trajectory in the phase space of a dynamical system such that trajectories starting in its vicinity asymptotically

approach it (stable limit cycle) or diverge from it (unstable limit cycle). It represents periodic behavior of the system.

Definition 3.26 (Neutrosophic Cognitive Map (NCM))**.** [9, 148, 185, 194, 207] *Neutrosophic Cognitive Map (NCM)* is a directed graph $\mathcal{G} = (C, E)$ whose vertices (concepts) are linked by edges (causal relationships) weighted by neutrosophic triples. Specifically:

(1) $C = \{C_1, C_2, \ldots, C_n\}$ is a finite set of n concepts representing variables, events, or processes in a system.

(2) $E \subseteq C \times C$ is the set of directed edges, where each edge (C_i, C_j) indicates a causal influence from C_i to C_j.

(3) Each edge (C_i, C_j) has a neutrosophic weight

$$W_{ij} = (T_{ij}, I_{ij}, F_{ij}), \quad \text{with } T_{ij} + I_{ij} + F_{ij} \leq 1,$$

where

$$T_{ij} \in [0,1] \quad \text{(truth or positive influence)}, \quad I_{ij} \in [0,1] \quad \text{(indeterminacy)},$$

$$F_{ij} \in [0,1] \quad \text{(falsity or negative influence)}.$$

Adjacency Matrix of an NCM. The adjacency matrix W of an NCM is a matrix whose (i,j)-th entry is $W_{ij} = (T_{ij}, I_{ij}, F_{ij})$. Each row-column entry thus encodes the neutrosophic weights from one concept to another.

State Vector. At time t, the system's state is given by a vector

$$A(t) = [\, a_1(t),\, a_2(t), \ldots, a_n(t)\,],$$

where each $a_i(t) \in [0,1]$ denotes the *activation level* of concept C_i at time t.

State Update Rule. The state evolves in discrete time. Given the adjacency matrix W, the new state $A(t+1)$ is computed by:

$$A(t+1) \;=\; \text{Threshold}\Big(A(t) \cdot W\Big),$$

where

$$\big(A(t) \cdot W\big)_j \;=\; \sum_{i=1}^{n} \big(T_{ij} \cdot a_i(t) \;-\; F_{ij} \cdot a_i(t) \;+\; I_{ij} \cdot a_i(t)\big),$$

and Threshold$(\cdot)$ re-scales or normalizes the result to remain within $[0,1]^n$.

Fixed Point and Limit Cycle. A *fixed point* is a state A^* where $A^* = \text{Threshold}(A^* \cdot W)$. A *limit cycle* is a finite sequence of states $\{\, A(t), A(t+1), \ldots, A(t+k)\}$ that repeats periodically.

Example 3.27 (NCM Illustration)**.** *Scenario:* Suppose we have three concepts related to a simplified economic model:

- C_1: Employment rate
- C_2: Investment level

- C_3: Consumer confidence

Let these concepts form an NCM with the following edges and neutrosophic weights:

$$W_{12} = (0.6,\, 0.2,\, 0.0), \quad W_{23} = (0.4,\, 0.3,\, 0.1), \quad W_{31} = (0.0,\, 0.2,\, 0.5),$$

and assume no other direct influences exist, so any missing edges have weight $(0, 0, 0)$.

Interpretation of Weights:

- $W_{12} = (0.6, 0.2, 0.0)$: If the employment rate (C_1) rises, it *positively* influences investment level (C_2) with a truth degree of 0.6. There's some uncertainty (0.2) about the relationship, and no negative influence.
- $W_{23} = (0.4, 0.3, 0.1)$: Investment level (C_2) tends to increase consumer confidence (C_3) but with moderate uncertainty and a small negative component (e.g., risk of inflation).
- $W_{31} = (0.0, 0.2, 0.5)$: Consumer confidence (C_3) might *negatively* impact employment rate if, for example, overconfidence leads to unstable spending (falsity = 0.5). There's also 0.2 uncertainty in this link.

State Update:

$$A(t) = [\, a_1(t),\, a_2(t),\, a_3(t)\,].$$

Then

$$A(t+1) = \text{Threshold}\Big(A(t)\,\cdot\,W\Big) = \text{Threshold}\Big(\big(\,X_1,\; X_2,\; X_3\big)\Big),$$

where, for instance,

$$X_2 = \sum_{i=1}^{3} \big(T_{i2} - F_{i2} + I_{i2}\big)\; a_i(t).$$

If we pick an initial state $A(0) = [0.5,\, 0.3,\, 0.8]$, we iteratively update $A(1),\, A(2), \ldots$ until the system stabilizes or settles into a cycle.

Why It's Useful:

- *Capturing Uncertainty*: Each weight includes truth, indeterminacy, and falsity—helpful when exact causal strengths are not fully known.
- *Modeling Complex Feedback Loops*: NCMs can capture cyclical influences (e.g., $C_1 \to C_2$ and $C_2 \to C_3$ and possibly $C_3 \to C_1$).
- *Possible Outcomes*: The model might converge to a fixed point (e.g., stable employment-investment-confidence levels) or oscillate if the feedback loops are strong or uncertain.

This example provides an illustration of how to construct and interpret an NCM in a simple economic context.

Definition 3.28 (HyperNeutrosophic Cognitive Map (HNCM))**.** A *HyperNeutrosophic Cognitive Map (HNCM)* is a generalization of a Neutrosophic Cognitive Map, in which each directed

edge is associated with a *set* of neutrosophic weights rather than a single neutrosophic triple. Formally, let

$$\mathcal{G} = (C, E)$$

be a directed graph where:

- $C = \{C_1, C_2, \ldots, C_n\}$ is a finite set of n concepts.
- $E \subseteq C \times C$ is a set of directed edges.

For each edge $(C_i, C_j) \in E$, the associated weight is a *HyperNeutrosophic Set*

$$W_{ij} \ \subseteq \ [0,1]^3,$$

where each element of W_{ij} is a triple $\big(T_k(C_i, C_j), I_k(C_i, C_j), F_k(C_i, C_j)\big)$ satisfying:

$$0 \ \le \ T_k(C_i, C_j) \ + \ I_k(C_i, C_j) \ + \ F_k(C_i, C_j) \ \le \ 3.$$

That is, for edge (C_i, C_j),

$$W_{ij} \ = \ \Big\{ \big(T_k(C_i, C_j), I_k(C_i, C_j), F_k(C_i, C_j)\big) \ \Big| \ k \in \mathcal{K}_{ij} \Big\} \ \subseteq \ [0,1]^3,$$

where $\mathcal{K}_{ij}$ is an index set representing multiple evaluations or sources of uncertainty for the causal relationship from C_i to C_j.

HNCM Adjacency Representation. The adjacency structure of an HNCM can be recorded in a matrix $\mathbf{W}$ whose (i,j)-th entry is the hyperneutrosophic set W_{ij}. That is,

$$\mathbf{W} = \big[\, W_{ij} \big], \quad W_{ij} \ \subseteq \ [0,1]^3.$$

State Vector and Update Rule. Let $A(t) = \big[a_1(t), a_2(t), \ldots, a_n(t)\big]$ be the state of the concepts at time t, where $a_i(t) \in [0,1]$. An HNCM typically requires an *aggregation operator* to combine the multiple neutrosophic triples in W_{ij}. One common approach is to define a function $\mathrm{Agg} : \mathcal{P}([0,1]^3) \to [0,1]^3$ that aggregates the set of triples in W_{ij} into a single effective triple:

$$\mathrm{Agg}\big(W_{ij}\big) = \big(\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij}\big),$$

where

$$\overline{T}_{ij} + \overline{I}_{ij} + \overline{F}_{ij} \ \le \ 3.$$

Once aggregated, the state update rule can be modeled analogously to a standard NCM, for example:

$$\big(A(t+1)\big)_j = \mathrm{Threshold}\Big(\sum_{i=1}^{n} \Big[\overline{T}_{ij}\, a_i(t) \ - \ \overline{F}_{ij}\, a_i(t) \ + \ \overline{I}_{ij}\, a_i(t)\Big]\Big),$$

where Threshold is a normalization (cf. [20, 36]) or clipping function (cf. [44]) ensuring $(A(t+1))_j \in [0,1]$. Alternative aggregation or update rules may be used depending on the application.

Definition 3.29 (n-SuperHyperNeutrosophic Cognitive Map (n-SHNCM))**.** An n-*SuperHyperNeutrosophic Cognitive Map* is a further generalization of a HyperNeutrosophic Cognitive Map, where each directed edge is associated with an *n-SuperHyperNeutrosophic Set* rather than a HyperNeutrosophic Set. Formally, let

$$\mathcal{G} = (C, E)$$

be a directed graph where:

- $C = \{C_1, C_2, \ldots, C_n\}$ is a finite set of n concepts.
- $E \subseteq C \times C$ is a set of directed edges.

For each edge $(C_i, C_j) \in E$, the associated weight is an *n-SuperHyperNeutrosophic Set*

$$W_{ij}^{(n)} : \tilde{\mathcal{P}}_n(\{(C_i, C_j)\}) \;\to\; \tilde{\mathcal{P}}_n\big([0,1]^3\big),$$

where $\tilde{\mathcal{P}}_n$ denotes the n-th nested family of non-empty subsets, as in the definition of n-SuperHyperNeutrosophic Sets. Concretely, $W_{ij}^{(n)}$ assigns to each $A \in \tilde{\mathcal{P}}_n(\{(C_i, C_j)\})$ a subset of $[0,1]^3$, such that:

$$0 \;\leq\; T + I + F \;\leq\; 3$$

for each triple (T, I, F) in that subset.

Intuitive Interpretation. The n-SuperHyperNeutrosophic Set on each edge (C_i, C_j) encapsulates not just multiple neutrosophic evaluations (k-indexed), but multiple levels (or layers) of hierarchical uncertainty. For instance, the first level might capture direct uncertainty from data, the second level might capture expert disagreements about that data, and so forth, up to n nested levels.

Adjacency Representation. The adjacency structure of an n-SHNCM can be encoded in an *n-level superhyperneutrosophic matrix*, with each entry $W_{ij}^{(n)}$ being an n-SuperHyperNeutrosophic Set that maps nested subsets to subsets of $[0,1]^3$.

State Vector and Update Rule. Let

$$A(t) = [\, a_1(t), a_2(t), \ldots, a_n(t) \,]$$

be the state vector at time t. To apply an update, one must first define a procedure to *collapse* or *aggregate* the n-SuperHyperNeutrosophic Set on each edge into an effective triple or small set of triples suitable for computing a concept's activation. For example, one might define:

$$\mathrm{Agg}_n\big(W_{ij}^{(n)}\big) \;\subseteq\; [0,1]^3$$

as an aggregation operator that extracts the relevant truth, indeterminacy, and falsity values from the hierarchical structure. Then, each concept's next state $(A(t+1))_j$ can be computed

using a generalized version of the NCM or HNCM update mechanism:

$$(A(t+1))_j = \text{Threshold}\Big(\sum_{i=1}^{n}\Big[\overline{T}_{ij}\,a_i(t) \;-\; \overline{F}_{ij}\,a_i(t) \;+\; \overline{I}_{ij}\,a_i(t)\Big]\Big),$$

where $\{\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij}\} \subseteq \text{Agg}_n\big(W_{ij}^{(n)}\big)$ and Threshold ensures the result remains in $[0, 1]$.

Key Properties of an n-SHNCM:

- *Hierarchical Uncertainty:* Multiple nested layers of indeterminacy or conflicting data are represented within each edge.
- *Complex Aggregation:* The user must specify an aggregation operator Agg_n to interpret the hierarchical structure for updating concept states.
- *Generalized Dynamics:* Similar to standard cognitive maps, the system may evolve to fixed points, limit cycles, or exhibit chaotic behavior, but now under deeper multi-level uncertainty.

Theorem 3.30 (Generalization Property of n-SuperHyperNeutrosophic Cognitive Maps)**.** *An* n-SuperHyperNeutrosophic Cognitive Map *(*n-SHNCM*) strictly generalizes both the HyperNeutrosophic Cognitive Map (HNCM) and the Neutrosophic Cognitive Map (NCM). Concretely:*

(1) *If $n = 0$, the n-SHNCM reduces to a standard Neutrosophic Cognitive Map.*

(2) *If $n = 1$, the n-SHNCM reduces to a HyperNeutrosophic Cognitive Map.*

For $n > 1$, it provides additional nesting of neutrosophic information, thereby generalizing both HNCMs ($n = 1$) and NCMs ($n = 0$).

Proof. We recall the following definitions:

- A *Neutrosophic Cognitive Map* (NCM) assigns, to each directed edge (C_i, C_j), exactly one neutrosophic triple (T_{ij}, I_{ij}, F_{ij}), with $T_{ij} + I_{ij} + F_{ij} \leq 1$.
- A *HyperNeutrosophic Cognitive Map* (HNCM) assigns, to each edge (C_i, C_j), a *set* of neutrosophic triples $W_{ij} \subseteq [0, 1]^3$. Each triple in that set must satisfy $T + I + F \leq 3$. Typically, after an *aggregation step*, one obtains an effective triple (or small set of triples) to compute causal influences.
- An *n-SuperHyperNeutrosophic Cognitive Map* (n-SHNCM) assigns, to each edge (C_i, C_j), an *n-SuperHyperNeutrosophic Set* of neutrosophic triples. This set is recursively or hierarchically defined up to n levels, capturing additional layers of uncertainty or contradictory evaluations.

Case $n = 0$: By definition, a 0-SuperHyperNeutrosophic Set for each edge is just a single neutrosophic triple $\big(T_{ij}, I_{ij}, F_{ij}\big)$. Hence each edge (C_i, C_j) holds precisely one triple. This matches exactly the data structure of a standard Neutrosophic Cognitive Map. Therefore, when $n = 0$, the n-SHNCM *coincides* with an NCM.

Case $n = 1$: When $n = 1$, each edge (C_i, C_j) is assigned a *HyperNeutrosophic Set* of neutrosophic triples rather than just one triple. This is precisely the definition of a HyperNeutrosophic Cognitive Map (HNCM). Hence for $n = 1$, the n-SHNCM *reduces* to the HNCM framework.

Case $n > 1$: For $n > 1$, an n-SuperHyperNeutrosophic Set is a *nested*, higher-order generalization of a HyperNeutrosophic Set. One obtains successively deeper layers of neutrosophic evaluations. The resulting map can capture more complex or multi-level contradictory opinions. This structure naturally subsumes the single-level sets of a HNCM ($n = 1$) and the single triple of an NCM ($n = 0$).

Hence, for each integer $n \geq 0$:

$$n = 0 \;\Rightarrow\; \text{standard NCM}, \quad n = 1 \;\Rightarrow\; \text{HNCM}, \quad n > 1 \;\Rightarrow\; \text{strict generalization beyond HNCM}.$$

Thus, an n-SHNCM indeed generalizes both HNCM ($n = 1$) and NCM ($n = 0$). □

Theorem 3.31 (Layer-by-Layer Aggregation Consistency). *Let* Agg_n *be an aggregation operator that collapses each n-SuperHyperNeutrosophic Set* $W_{ij}^{(n)}$ *into finitely many (or one) representative triple(s). Suppose for each edge* (C_i, C_j)*,*

$$\mathrm{Agg}_n\big(W_{ij}^{(n)}\big) \subseteq [0,1]^3,$$

and for every triple (T, I, F) *in the output, we have* $T + I + F \leq 3$*. If* Agg_n *is* layer-wise consistent*, i.e. it respects the nested structure in* $\tilde{\mathcal{P}}_n\big([0,1]^3\big)$ *at each level, then the final aggregated map is well-defined for an n-SHNCM.*

Statement. Layer-wise consistency means that if $V_{ij}^{(k)} \subset \tilde{\mathcal{P}}_k([0,1]^3)$ *is a k-fold subset at level k, and* $V_{ij}^{(k+1)}$ *extends it at level* $(k+1)$*, then*

$$\mathrm{Agg}_{k+1}(V_{ij}^{(k+1)}) \quad \textit{coincides or refines} \quad \mathrm{Agg}_k(V_{ij}^{(k)}),$$

ensuring no contradictions among nested layers. Under this assumption, the aggregator Agg_n *produces a unique or consistent triple set at the final output, guaranteeing a well-defined n-SHNCM adjacency representation.*

Proof. Consider the nested sets:

$$V^{(1)} \subseteq V^{(2)} \subseteq \cdots \subseteq V^{(n)},$$

where each $V^{(k)} \in \tilde{\mathcal{P}}_k([0,1]^3)$. By hypothesis, Agg_n composes Agg_k layer by layer. Specifically, $\mathrm{Agg}_{k+1}\big(V^{(k+1)}\big)$, restricted to level k, must match $\mathrm{Agg}_k\big(V^{(k)}\big)$. This compositional property ensures that the final aggregator output at level n is independent of the path chosen to aggregate intermediate subsets. Hence, the aggregator is well-defined across all edges (C_i, C_j).

As each triple (T, I, F) satisfies $T + I + F \le 3$, the neutrosophic constraints remain intact, thereby giving a well-defined final adjacency representation for the n-SHNCM. □

Theorem 3.32 (Fixed Point under Contractive Aggregation). *Let an n-SHNCM have a state update rule*

$$A(t+1) = F_n\big(A(t)\big),$$

where $F_n : [0,1]^n \to [0,1]^n$ is formed by aggregating each n-SuperHyperNeutrosophic edge set $W_{ij}^{(n)}$ and summing influences. Assume there exists a metric $d(\cdot,\cdot)$ on $[0,1]^n$ such that F_n is a strict contraction, i.e. for all $A, B \in [0,1]^n$:

$$d\big(F_n(A),\, F_n(B)\big) \;\le\; \lambda\, d(A,B), \quad \textit{for some constant } 0 < \lambda < 1.$$

Then there exists a unique fixed point $A^* \in [0,1]^n$ such that

$$F_n(A^*) = A^*.$$

Proof. This is an application of the Banach Fixed Point Theorem (also known as the Contraction Mapping Principle). Since F_n is defined on the complete metric space $([0,1]^n, d)$ and satisfies $d(F_n(A), F_n(B)) \le \lambda\, d(A,B)$ with $\lambda < 1$, there is a unique fixed point A^* satisfying $A^* = F_n(A^*)$. Existence follows by iterative updates from any initial state, and uniqueness follows because a strict contraction cannot admit two distinct fixed points. □

Theorem 3.33 (Boundedness of $A(t)$ for n-SHNCM). *In an n-SHNCM, let $A(t) \in [0,1]^n$ evolve via any aggregator-based update rule. Then for all times $t \ge 0$, we have $A(t) \in [0,1]^n$. In other words, the state remains in the unit hypercube $[0,1]^n$ regardless of the complexity or depth n of the superhyperneutrosophic edges.*

Proof. By definition, each aggregator $\mathrm{Agg}_n\big(W_{ij}^{(n)}\big)$ returns neutrosophic triples (T, I, F) with $T + I + F \le 3$ and each $T, I, F \in [0,1]$. The update for each concept C_j is typically:

$$(A(t+1))_j = \mathrm{Threshold}\Big(\sum_{i=1}^{n} \big[\overline{T}_{ij}\, a_i(t) - \overline{F}_{ij}\, a_i(t) + \overline{I}_{ij}\, a_i(t)\big]\Big),$$

where Threshold is some normalization or clipping that maps real values into $[0,1]$. Since $a_i(t) \in [0,1]$ and $\overline{T}_{ij}, \overline{F}_{ij}, \overline{I}_{ij} \in [0,1]$, the inner expression is bounded (in, say, $[-n, +n]$). The Threshold function ensures $(A(t+1))_j \in [0,1]$. Therefore, $A(t+1) \in [0,1]^n$, and by induction on t (starting from some $A(0) \in [0,1]^n$), all future states remain in $[0,1]^n$. □

A classifier is a function or algorithm that assigns input data to predefined categories or classes based on learned patterns or rules from training data [12, 28, 234]. A Neutrosophic Classifier leverages neutrosophic logic to address uncertainty by assigning truth, indeterminacy, and falsity degrees for classification tasks [15, 16, 71, 285].

A related concept is the Fuzzy Classifier, which uses fuzzy logic to handle uncertainty by assigning membership degrees to classes, enabling flexible and imprecise decision boundaries [39, 165, 208, 301]. Additionally, the Intuitionistic Fuzzy Classifier is another known approach in this context [154, 265, 273].

This section introduces the HyperNeutrosophic Classifier and the n-SuperHyperNeutrosophic Classifier, which extend the Neutrosophic Classifier framework to higher levels of complexity and abstraction.

Definition 3.34 (Supervised Learning)**.** [40, 55, 59, 119] In general, Supervised Learning is a method in machine learning that involves learning a relationship or mapping between a set of input variables X and an output variable Y based on labeled training data. The goal is to construct a model f, which minimizes the risk and accurately predicts the output for unseen data. The training dataset consists of pairs (x_i, y_i), where $x_i \in X$ and $y_i \in Y$. Using the principle of risk minimization, the model f is optimized to generalize well to new data. Once trained, the model can be applied to infer outputs for new, unlabeled inputs.

Notation 2. *All definitions assume a supervised learning context in which we have:*

$$\textit{Training set } \mathcal{D} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\},$$

where $x_i \in \mathbb{R}^d$ *(feature space), and* $y_i \in \mathcal{C} = \{c_1, c_2, \ldots, c_K\}$ *is the set of class labels.*

Definition 3.35 (Neutrosophic Classifier)**.** Let $\mathcal{X} = \mathbb{R}^d$ be the feature space, and let $\mathcal{C} = \{c_1, c_2, \ldots, c_K\}$ be a finite set of classes. A *Neutrosophic Classifier* is a function

$$\mathcal{N} : \mathcal{X} \ \to \ [0,1]^3 \times \cdots \times [0,1]^3 \ (K \text{ times}),$$

so that for each input $x \in \mathcal{X}$, the classifier outputs a K-tuple of neutrosophic membership triples:

$$\mathcal{N}(x) \ = \ \big((T_1(x), I_1(x), F_1(x)), \ \ldots, \ (T_K(x), I_K(x), F_K(x))\big),$$

where each triple $(T_j(x), I_j(x), F_j(x)) \in [0,1]^3$ satisfies

$$T_j(x) + I_j(x) + F_j(x) \ \leq \ 3.$$

Intuitively, $T_j(x)$ represents the degree of truth-membership (confidence that x is in class c_j), $I_j(x)$ the degree of indeterminacy, and $F_j(x)$ the degree of falsity.

Classification Decision: Typically, one might apply a *defuzzification* or *neutrosophic decision* function to map the neutrosophic outputs to a crisp label in $\mathcal{C}$. For example:

$$\hat{y}(x) = \arg\max_{1 \le j \le K} \Big(T_j(x) - F_j(x) \Big),$$

or any other decision rule that accounts for (T_j, I_j, F_j).

Definition 3.36 (HyperNeutrosophic Classifier)**.** Let $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{C} = \{c_1, \ldots, c_K\}$. A *HyperNeutrosophic Classifier* is a function

$$\mathcal{HN} : \mathcal{X} \;\to\; \big(\mathcal{P}([0,1]^3)\big)^K,$$

where for each $x \in \mathcal{X}$ and for each class c_j,

$$\mathcal{HN}(x)_j \;=\; W_j(x) \;\subseteq\; [0,1]^3,$$

is a *HyperNeutrosophic Set* of possible triples $\{(T_k, I_k, F_k)\}$. Each triple (T_k, I_k, F_k) within $W_j(x)$ must satisfy

$$T_k + I_k + F_k \;\le\; 3.$$

Rather than assigning one neutrosophic triple to class c_j, the HyperNeutrosophic Classifier assigns a *nonempty subset* of $[0,1]^3$. Different elements in this subset might represent multiple experts' opinions, different data sources, or uncertain/conflicting evaluations of the membership degrees for class c_j.

Decision Rule: One commonly aggregates each set $W_j(x)$ into a single representative triple, e.g.

$$\overline{(T,I,F)}_j(x) = \mathrm{Agg}\Big(W_j(x)\Big),$$

then apply a neutrosophic-based decision rule, such as

$$\hat{y}(x) = \arg\max_{1 \le j \le K} \big(\overline{T}_j(x) - \overline{F}_j(x)\big).$$

Definition 3.37 (n-SuperHyperNeutrosophic Classifier)**.** Let $\mathcal{X} = \mathbb{R}^d$ and $\mathcal{C} = \{c_1, \ldots, c_K\}$.

An *n-SuperHyperNeutrosophic Classifier* is a function

$$\mathcal{SHN}^{(n)} : \mathcal{X} \;\to\; \Big(\tilde{\mathcal{P}}_n\big([0,1]^3\big)\Big)^K,$$

where $\tilde{\mathcal{P}}_n([0,1]^3)$ denotes the n-th nested family of non-empty subsets of the unit cube $[0,1]^3$. Concretely, for each $x \in \mathcal{X}$ and class c_j,

$$\mathcal{SHN}^{(n)}(x)_j \;=\; \mathcal{W}_j^{(n)}(x),$$

where $\mathcal{W}_j^{(n)}(x) \in \tilde{\mathcal{P}}_n([0,1]^3)$. Each nested membership structure satisfies the constraint

$$\forall (T,I,F) \in \mathcal{W}_j^{(n)}(x) \;:\quad T + I + F \;\le\; 3.$$

At level $n = 0$, $\mathcal{W}_j^{(0)}(x)$ is simply a single triple (T, I, F), matching a standard Neutrosophic Classifier. At level $n = 1$, $\mathcal{W}_j^{(1)}(x) \subseteq [0,1]^3$ is a *HyperNeutrosophic set*, i.e. multiple triples.

For $n \geq 2$, one obtains hierarchically nested sets-of-sets, capturing multi-level or multi-source uncertainty at each depth.

Decision Step: One typically defines an aggregation operator $\mathrm{Agg}_n : \tilde{\mathcal{P}}_n([0,1]^3) \to [0,1]^3$ to collapse each nested membership structure to a single triple. Then the classification decision can be performed via a neutrosophic-based rule on these aggregated triples.

Theorem 3.38 (Reduction Property of n-SuperHyperNeutrosophic Classifier)**.** *An n-SuperHyperNeutrosophic Classifier reduces to a HyperNeutrosophic Classifier if $n = 1$, and reduces to a standard Neutrosophic Classifier if $n = 0$.*

Proof. By definition, $\tilde{\mathcal{P}}_0([0,1]^3)$ is just a single triple (T, I, F). Thus, if $n = 0$, $\mathcal{SHN}^{(0)}$ assigns exactly one triple to each class, i.e. a Neutrosophic Classifier. If $n = 1$, $\tilde{\mathcal{P}}_1([0,1]^3) = \mathcal{P}([0,1]^3)$ is a set of one or more triples in $[0,1]^3$, forming a HyperNeutrosophic set. Hence $\mathcal{SHN}^{(1)}$ becomes a HyperNeutrosophic Classifier. For $n > 1$, each class membership is an n-th nested structure, strictly more general than $n = 1$ or $n = 0$. □

Theorem 3.39 (Well-definedness of $\mathcal{SHN}^{(n)}$)**.** *Let $\mathcal{SHN}^{(n)}$ be an n-SuperHyperNeutrosophic Classifier. Suppose for each class c_j, $\mathcal{W}_j^{(n)}(x) \in \tilde{\mathcal{P}}_n([0,1]^3)$ is generated via a function*

$$\Phi_j^{(n)} : \mathcal{X} \to \tilde{\mathcal{P}}_n([0,1]^3),$$

which respects the condition $T + I + F \leq 3$. Then $\mathcal{SHN}^{(n)}$ is well-defined for all $x \in \mathcal{X}$.

Proof. We must check that each $\mathcal{W}_j^{(n)}(x)$ is indeed in $\tilde{\mathcal{P}}_n([0,1]^3)$. By assumption, $\Phi_j^{(n)}$ constructs an element of the n-th nested power set. Moreover, each triple within the structure satisfies $T+I+F \leq 3$. Since $j \in \{1, \dots, K\}$ is finite, the classifier output is a K-tuple of valid n-SuperHyperNeutrosophic sets. Thus, $\mathcal{SHN}^{(n)}$ is properly defined on the entire domain $\mathcal{X}$. □

Theorem 3.40 (Continuity of a Parametric $\mathcal{SHN}^{(n)}$)**.** *Suppose $\mathcal{SHN}^{(n)}$ depends on parameters Θ, i.e. each $\mathcal{W}_j^{(n)}(x)$ is generated by a continuous function of (x, Θ) into $\tilde{\mathcal{P}}_n([0,1]^3$. If the aggregator and set-construction steps are all continuous with respect to Θ and x, then $\mathcal{SHN}^{(n)}$ is a continuous map from $\mathcal{X} \times \Theta$ into $\left(\tilde{\mathcal{P}}_n([0,1]^3)\right)^K$.*

Proof. Each step in producing $\mathcal{W}_j^{(n)}(x)$ is assumed continuous with respect to the real parameters Θ and the input x. The nested structure $\tilde{\mathcal{P}}_n([0,1]^3)$ can be embedded or represented in a suitable topological space (e.g. via canonical encodings or representing each level's subsets). Under a standard product topology, continuity follows by composition of continuous maps at each stage. Therefore, $\mathcal{SHN}^{(n)}(x; \Theta)$ is continuous in both arguments. □

Theorem 3.41 (Fixed-Structure Theorem for Classifier Consistency). *Consider a training set* $\{(x_i, y_i)\}$. *Suppose the classifier* $\mathcal{SHN}^{(n)}$ *uses a fixed, non-trainable aggregator* Agg_n *that maps each* $\mathcal{W}_j^{(n)}(x)$ *to a single triple* $\overline{(T, I, F)}_j(x)$. *If* Agg_n *and the mapping from input to* $\mathcal{W}_j^{(n)}(x)$ *remain unchanged during training, then the classification boundaries are determined by the aggregator result. Specifically, if* $\hat{y}(x) = \arg\max_j(\overline{T}_j(x) - \overline{F}_j(x))$, *any parameter updates that do not alter* $\mathcal{W}_j^{(n)}(x)$ *or aggregator logic cannot change the decision boundary.*

Proof. Because the aggregator Agg_n and the membership structure $\mathcal{W}_j^{(n)}(x)$ are assumed fixed, none of the computed triple $\overline{(T, I, F)}_j(x)$ can change. Hence, for all x, the final numeric score $\overline{T}_j(x) - \overline{F}_j(x)$ is invariant. Therefore, the classification boundary (i.e. the set of $x \in \mathcal{X}$ where $\max_j(\overline{T}_j(x) - \overline{F}_j(x))$ is tied or changes among classes) remains the same. Thus, no parameter modifications that do not affect $\mathcal{W}_j^{(n)}$ or aggregator logic can alter the classifier's decision function. □

Theorem 3.42 (Universal Approximation under Suitable Encodings). *Let* $\mathcal{SHN}^{(n)}$ *be a parametric n-SuperHyperNeutrosophic classifier, which encodes each n-SuperHyperNeutrosophic set into a finite-dimensional vector (through an appropriate embedding) and then applies a universal approximator (e.g. a multilayer perceptron). Suppose the embedding and aggregator are sufficiently flexible to represent or approximate any continuous function* $\mathcal{X} \to \left(\tilde{\mathcal{P}}_n([0,1]^3)\right)^K$. *Then, for any continuous target classification function* $f^*(x)$ *(mapping* x *to some ideal membership structure in* $\tilde{\mathcal{P}}_n([0,1]^3)$*), there exists a sequence of parameter settings* Θ_m *such that*

$$\lim_{m\to\infty} \left\| \mathcal{SHN}^{(n)}(x; \Theta_m) - f^*(x) \right\| = 0,$$

uniformly on compact subsets of $\mathcal{X}$. *Hence,* $\mathcal{SHN}^{(n)}$ *is a universal approximator in the domain of nested neutrosophic set-based classification.*

Proof. The universal approximation argument proceeds by noting that each element of $\tilde{\mathcal{P}}_n([0,1]^3)$ can be encoded into a finite (though possibly large) dimensional representation (e.g., by enumerating or sampling from the nested membership sets). A sufficiently expressive neural network or parametric system can approximate any continuous mapping from $\mathcal{X} \to \mathcal{Z}$ for $\mathcal{Z} \subset \mathbb{R}^M$. Composing this neural net with a suitable *decoding* step that reconstructs or interprets the aggregated sets ensures that the entire map can approximate any continuous target function f^*. The details rely on standard universal approximation theorems plus a consistent encoding/decoding scheme for the nested sets. Convergence in the sup norm (or uniform metric) follows from classical results in neural net approximation theory. □

A Neutrosophic Triplet is an ordered triple that adheres to specific neutral and anti-properties with respect to a binary operation [10, 221, 222, 257, 259]. Related concepts, such as Neutrosophic Duplets, are also recognized in the literature [149, 179, 296]. This framework has been further extended to encompass the HyperNeutrosophic Triplet and the n-SuperHyperNeutrosophic Triplet.

Definition 3.43 (Neutrosophic Triplet)**.** [257] Let $(N, \star)$ be a nonempty set N with a binary operation $\star : N \times N \to N$. For an element $a \in N$, a *neutrosophic triplet* is an ordered triple $(a, \mathrm{neut}(a), \mathrm{anti}(a))$ such that:

$$a \star \mathrm{neut}(a) \;=\; \mathrm{neut}(a) \star a \;=\; a, \quad \text{and} \quad a \star \mathrm{anti}(a) \;=\; \mathrm{anti}(a) \star a \;=\; \mathrm{neut}(a).$$

The elements a, $\mathrm{neut}(a)$, and $\mathrm{anti}(a)$ are said to form a neutrosophic triplet. In this context:

- $\mathrm{neut}(a)$ is called a *neutral of* a (which replaces or generalizes an identity-like element but only relative to a).
- $\mathrm{anti}(a)$ is called an *anti of* a (which replaces or generalizes an inverse-like element but only relative to a).

Definition 3.44 (Neutrosophic Triplet Group)**.** A *Neutrosophic Triplet Group (NTG)* is a pair $(N, \star)$ such that:

(1) *Closure:* For any $x, y \in N$, we have $x \star y \in N$.
(2) *Associativity:* For all $x, y, z \in N$, $(x \star y) \star z = x \star (y \star z)$.
(3) *Existence of Neutrosophic Triplets:* For every $a \in N$, there exist $\mathrm{neut}(a), \mathrm{anti}(a) \in N$ such that

$$a \star \mathrm{neut}(a) = \mathrm{neut}(a) \star a = a, \quad a \star \mathrm{anti}(a) = \mathrm{anti}(a) \star a = \mathrm{neut}(a).$$

We emphasize that $\mathrm{neut}(a)$ replaces the notion of a group identity for the specific element a, and $\mathrm{anti}(a)$ replaces the notion of an inverse for a. Unlike a classical group, these neutral and anti elements can vary with a.

Example 3.45. Consider $(Z_6, \times_6)$, where $\times_6$ is multiplication modulo 6. We observe:

$$2 \times_6 4 \;=\; 8 \equiv 2 \pmod 6, \quad 2 \times_6 2 \;=\; 4 \pmod 6.$$

Hence for $a = 2$:

$$\mathrm{neut}(2) = 4, \quad \mathrm{anti}(2) = 2,$$

satisfying

$$2 \star 4 = 4 \star 2 = 2, \quad 2 \star 2 = 4, \quad 4 \star 4 = 4.$$

Thus $(2,4,2)$ is a neutrosophic triplet. Checking associativity and closure reveals $(Z_6, \times_6)$ is not a classical group, yet it has valid neutrosophic triplets for certain elements. If each element admits such triplets, it forms an NTG (modulo verifying all conditions).

Definition 3.46 (Commutative NTG)**.** An NTG $(N, \star)$ is *commutative* if $x \star y = y \star x$ for all $x, y \in N$.

Definition 3.47 (HyperNeutrosophic Triplet Group)**.** A *HyperNeutrosophic Triplet Group (HNTG)* is a pair $(N, \star)$ such that:

(1) *Closure & Associativity:* For all $x, y \in N$, $x \star y \in N$, and $\star$ is associative.

(2) *HyperNeutrosophic Triplets:* For each $a \in N$, there is a *set of neutrals* $\mathrm{NeutSet}(a) \subseteq N$ and a *set of antis* $\mathrm{AntiSet}(a) \subseteq N$, such that for every $b \in \mathrm{NeutSet}(a)$ and $c \in \mathrm{AntiSet}(a)$,

$$a \star b = b \star a = a, \quad \text{and} \quad a \star c = c \star a = b.$$

In other words, each element a has multiple possible pairs (b, c) forming hyperneutrosophic triplets (a, b, c).

Remark 3.48. In a HyperNeutrosophic Triplet Group, each a can have infinitely many neutrals $\mathrm{NeutSet}(a)$ and infinitely many antis $\mathrm{AntiSet}(a)$. This is an extension of the single-triple idea where we replace $\mathrm{neut}(a)$ by a set of possibilities and $\mathrm{anti}(a)$ by another set.

Example 3.49. Let $(N, \star)$ be the same set as in a Neutrosophic Triplet Group example, but suppose for each element a, we define a set of neutral candidates $\mathrm{NeutSet}(a) \subseteq N$ and a set of anti candidates $\mathrm{AntiSet}(a) \subseteq N$. As long as for each $b \in \mathrm{NeutSet}(a)$ and $c \in \mathrm{AntiSet}(a)$ the required conditions hold, we get a valid HNTG.

Definition 3.50 (***n***-SuperHyperNeutrosophic Triplet Group)**.** Let n be a nonnegative integer. An *n-SuperHyperNeutrosophic Triplet Group (n-SHNTG)* is a pair $(N, \star)$ such that:

(1) *Closure & Associativity:* $\star$ is associative and closed on N.

(2) *Nested Triplets Up to Level n:* For each $a \in N$, we have an n-fold nested family of possible neutrals and antis. More explicitly, at level $k \leq n$, we define

$$\mathrm{NeutSet}^{(k)}(a) \subseteq N, \quad \mathrm{AntiSet}^{(k)}(a) \subseteq N.$$

At each level k, for every pair (b, c) with $b \in \mathrm{NeutSet}^{(k)}(a)$ and $c \in \mathrm{AntiSet}^{(k)}(a)$, the triple (a, b, c) satisfies

$$a \star b = b \star a = a, \quad a \star c = c \star a = b.$$

Additionally, the levels are *nested* in the sense that $\mathrm{NeutSet}^{(k+1)}(a)$ refines or extends $\mathrm{NeutSet}^{(k)}(a)$ and similarly for $\mathrm{AntiSet}^{(k)}(a)$.

Theorem 3.51 (Generalization Property)**.** *Every n-SHNTG with $n = 0$ reduces to a Neutrosophic Triplet Group (NTG), and every n-SHNTG with $n = 1$ reduces to a HyperNeutrosophic Triplet Group (HNTG).*

Proof. If $n = 0$, we allow no hyper-sets of neutrals or antis—only single elements. Hence the definition collapses exactly to a Neutrosophic Triplet Group: each a has a single neut(a) and single anti(a). If $n = 1$, for each a we define $\mathrm{NeutSet}^{(1)}(a)$, $\mathrm{AntiSet}^{(1)}(a)$ as sets of neutrals and antis, recovering the HNTG definition from Definition 3.47. □

Theorem 3.52 (Reduction Homomorphism)**.** *Let $\rho_{n\to m}$ be a surjective map from the n-SuperHyperNeutrosophic structure to an m-SuperHyperNeutrosophic structure, with $m < n$. Suppose*

$$\rho_{n\to m}(a \star b) = \rho_{n\to m}(a) \underline{\star}\, \rho_{n\to m}(b),$$

and similarly for all nest levels of neutrals and antis. Then $(N, \star)$ at level n reduces to (or homomorphically maps onto) the (m)-SHNTG structure.

Proof. We define $\rho_{n\to m}$ to *flatten* or *select subsets* from the n-level hyperstructures. If $\rho_{n\to m}$ respects the binary operation $\star$ (i.e., is an algebraic homomorphism) and commutes with the nested neutrals and antis at each level, the resulting image is a valid m-SHNTG. Details parallel standard homomorphism arguments in universal algebra but adapted to nested triplets. □

Theorem 3.53 (Existence of Trivial Triplets at Each Level)**.** *In an n-SHNTG, for every idempotent element x (i.e. $x \star x = x$) at any level $k \le n$, the triple (x, x, x) forms a trivial (hyper)neutrosophic triplet (or nested family thereof).*

Proof. If $x \star x = x$, then for each $k \le n$, one can place x in $\mathrm{NeutSet}^{(k)}(x)$ and $\mathrm{AntiSet}^{(k)}(x)$, satisfying $x \star x = x$ and $x \star x = x$. □

Theorem 3.54 (Associativity Preservation)**.** *In a HyperNeutrosophic or n-SHNTG, associativity is a global property that must hold for all elements, not just for those forming a single triplet. Consequently, modifying the set of possible neutrals/antis for certain elements must preserve associativity across the entire structure.*

Proof. Follows from standard arguments in universal algebra: The binary operation $\star$ must be globally associative, i.e. $(x \star y) \star z = x \star (y \star z)$ for every $x, y, z \in N$. Defining or modifying hyperneutrosophic sets $\mathrm{NeutSet}^{(k)}(x)$ or $\mathrm{AntiSet}^{(k)}(x)$ does not remove the requirement that $\star$ remains associative on all of N. □

4. Additional Result: Hyperfuzzy Extension

In this section, we explore extensions based on Hyperfuzzy and Superhyperfuzzy concepts rather than HyperNeutrosophic sets.

4.1. *Neuro-Hyperfuzzy System*

A Neuro-Fuzzy System integrates the learning capabilities of neural networks with the reasoning mechanisms of fuzzy logic, enabling effective decision-making in uncertain environments [41, 42, 78, 125, 136, 156, 158]. This hybrid approach leverages the strengths of both neural and fuzzy systems, making it suitable for a wide range of applications. Additionally, a related concept, the Neuro-Neutrosophic System, has been discussed in the literature [89, 271].

In this subsection, we examine the *Neuro-Hyperfuzzy System* and its extension, the *Neuro-Superhyperfuzzy System*. Definitions and relevant details are provided below.

Definition 4.1 (Neuro-Fuzzy System)**.** (cf. [78, 125, 156]) A *Neuro-Fuzzy System* (NFS) is a hybrid intelligent system that integrates fuzzy logic-based reasoning with the learning capabilities of neural networks. Formally, it is represented as a tuple:

$$\mathcal{N} = (\mathbb{X}, \mathbb{Y}, F, R, \mathcal{L}),$$

where:

- $\mathbb{X} \subseteq \mathbb{R}^n$: The input space, where n is the number of input variables.
- $\mathbb{Y} \subseteq \mathbb{R}^m$: The output space, where m is the number of output variables.
- $F = \{f_1, f_2, \ldots, f_p\}$: A set of fuzzy membership functions defined on $\mathbb{X}$.
- $R = \{r_1, r_2, \ldots, r_q\}$: A set of fuzzy rules, each of the form:

  $$r_k : \text{IF } \bigwedge_{i=1}^{n} (x_i \text{ is } \mu_{ki}) \text{ THEN } \bigwedge_{j=1}^{m} (y_j \text{ is } \nu_{kj}),$$

  where μ_{ki} and ν_{kj} are fuzzy sets for the inputs and outputs, respectively.
- $\mathcal{L}$: A learning algorithm that adjusts F and R based on training data $T = \{(x_t, y_t)\}_{t=1}^{N}$, where $x_t \in \mathbb{X}$ and $y_t \in \mathbb{Y}$.

Definition 4.2 (System Structure of Neuro-Fuzzy System)**.** The architecture of an NFS can be represented as a multi-layer feedforward network consisting of the following layers:

(1) *Input Layer:* Directly represents the input vector $x = (x_1, x_2, \ldots, x_n)$.

(2) *Fuzzification Layer:* Transforms crisp inputs into fuzzy values using membership functions:

$$\mu_{ki}(x_i) = f_{ki}(x_i), \quad \forall k, i.$$

(3) *Rule Layer:* Computes the firing strength of each fuzzy rule r_k as:

$$\text{Activation: } A_k = \bigwedge_{i=1}^{n} \mu_{ki}(x_i),$$

where $\bigwedge$ is a t-norm (e.g., the minimum operator).

(4) *Aggregation Layer:* Aggregates the outputs of all rules using:

$$\nu(y) = \bigvee_{k=1}^{q} \left(A_k \cdot \nu_k(y)\right),$$

where $\bigvee$ is a t-conorm (e.g., the maximum operator), and $\nu_k(y)$ is the output fuzzy set for rule r_k.

(5) *Defuzzification Layer:* Converts the aggregated fuzzy output into a crisp value:

$$y = \frac{\int_{y \in \mathbb{Y}} y \cdot \nu(y)\, dy}{\int_{y \in \mathbb{Y}} \nu(y)\, dy}.$$

Definition 4.3 (Learning Algorithm)**.** The learning process optimizes the parameters of F and R to minimize a loss function:

$$\mathcal{L}(\Theta) = \frac{1}{N} \sum_{t=1}^{N} \|y_t - \hat{y}(x_t; \Theta)\|^2,$$

where:

- Θ: The set of all parameters, including fuzzy membership function parameters and rule weights.
- $\hat{y}(x_t; \Theta)$: The output of the NFS for input x_t under the current parameter set Θ.

Gradient-based methods or heuristic approaches are employed to adjust Θ iteratively.

Definition 4.4 (Constraints and Interpretability)**.** To ensure interpretability and consistency:

- Fuzzy membership functions μ_{ki} must satisfy overlap constraints (e.g., intersections at membership degree 0.5).
- Rules R should be mutually non-contradictory and logically consistent.

Definition 4.5 (Neuro-Hyperfuzzy System (NHFS))**.** A *Neuro-Hyperfuzzy System* (NHFS) is a hybrid framework integrating a neural network architecture with hyperfuzzy-based membership functions and rules. Formally, we define it as a 5-tuple

$$\mathcal{N}_H = \left(\mathbb{X}, \mathbb{Y}, \tilde{F}, \tilde{R}, \mathcal{L}_H\right),$$

where:

- $\mathbb{X} \subseteq \mathbb{R}^n$: the input space ($n$ real-valued features).
- $\mathbb{Y} \subseteq \mathbb{R}^m$: the output space ($m$ real-valued target variables).

- $\tilde{F} = \{\tilde{f}_1, \ldots, \tilde{f}_p\}$: a set of *hyperfuzzy membership functions*, each

$$\tilde{f}_k : \mathbb{X} \to \tilde{P}([0,1]),$$

  meaning for every input $x \in \mathbb{X}$, $\tilde{f}_k(x) \subseteq [0,1]$ is a (non-empty) set of membership values.

- $\tilde{R} = \{\tilde{r}_1, \ldots, \tilde{r}_q\}$: a set of *hyperfuzzy rules*. Each rule $\tilde{r}_k$ has the form

$$\tilde{r}_k : \text{ IF } \bigwedge_{i=1}^{n} \big(x_i \text{ is } \tilde{\mu}_{ki}\big) \text{ THEN } \bigwedge_{j=1}^{m} \big(y_j \text{ is } \tilde{\nu}_{kj}\big),$$

  where $\tilde{\mu}_{ki}, \tilde{\nu}_{kj}$ are hyperfuzzy membership functions applied to inputs x_i and outputs y_j, respectively.

- $\mathcal{L}_H$: a *learning algorithm* that updates $\tilde{F}$ and $\tilde{R}$ based on training data $\{(x_t, y_t)\}_{t=1}^{N}$, aiming to minimize a specified loss function (e.g., MSE) while accommodating the hyperfuzzy membership framework.

Definition 4.6 (Neuro-n-SuperHyperfuzzy System (NSHFS))**.** Let $\tilde{\mathcal{P}}_n([0,1])$ be the n-SuperHyperfuzzy extension of $[0,1]$. A *Neuro-n-SuperHyperfuzzy System* extends the Neuro-Hyperfuzzy System to n nested levels of hyperfuzzy membership:

$$\mathcal{N}_{SH,n} = \Big(\mathbb{X}, \mathbb{Y}, \tilde{F}_n, \tilde{R}_n, \mathcal{L}_{H,n}\Big),$$

where:

- $\tilde{F}_n = \{\tilde{f}_{n,1}, \ldots, \tilde{f}_{n,p}\}$, each

$$\tilde{f}_{n,k} : \mathbb{X} \to \tilde{\mathcal{P}}_n([0,1]),$$

  describes membership via n-level nested sets in $[0,1]$.

- $\tilde{R}_n = \{\tilde{r}_{n,1}, \ldots, \tilde{r}_{n,q}\}$, each rule $\tilde{r}_{n,k}$ is:

$$\tilde{r}_{n,k} : \text{IF } \bigwedge_{i=1}^{n} \big(x_i \text{ is } \tilde{\mu}_{n,ki}\big) \text{ THEN } \bigwedge_{j=1}^{m} \big(y_j \text{ is } \tilde{\nu}_{n,kj}\big).$$

  Here, $\tilde{\mu}_{n,ki}, \tilde{\nu}_{n,kj}$ are n-SuperHyperfuzzy membership functions.

- $\mathcal{L}_{H,n}$ is a *learning algorithm* adapted to handle the n-SuperHyperfuzzy membership structure. It optimizes both $\tilde{F}_n$ and $\tilde{R}_n$ using training data under higher-order uncertainties.

Theorem 4.7 (Universal Approximation for Neuro-Hyperfuzzy Systems)**.** *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be a compact set, and let $f : \mathbb{X} \to \mathbb{R}^m$ be a continuous function. Then, for every $\varepsilon > 0$, there exists a Neuro-Hyperfuzzy System*

$$\mathcal{N}_H = (\mathbb{X}, \mathbb{Y}, \tilde{F}, \tilde{R}, \mathcal{L}_H)$$

such that for all $x \in \mathbb{X}$,

$$\|\widehat{f}(x) - f(x)\| < \varepsilon,$$

where $\widehat{f}(x)$ is the NHFS output.

Proof. Step 1: Discretization of the Input Space. Since $\mathbb{X}$ is compact in $\mathbb{R}^n$, we can construct a finite covering by hyper-rectangles or grid points $\{x^\ell\}_{\ell=1}^L$ such that each point of $\mathbb{X}$ lies within δ-distance of at least one x^ℓ. By the uniform continuity of f on a compact set, there exists $\delta > 0$ ensuring $|f(x) - f(x^\ell)| < \varepsilon/2$ whenever $\|x - x^\ell\| < \delta$.

Step 2: Construction of Hyperfuzzy Membership Functions. For each x^ℓ, define a hyperfuzzy membership function

$$\tilde{f}_\ell : \mathbb{X} \to \tilde{P}([0,1])$$

such that: 1. If x is within δ-distance of x^ℓ, $\tilde{f}_\ell(x)$ includes a subset of membership degrees near 1 (e.g., $[0.8, 1] \subseteq \tilde{f}_\ell(x)$). 2. Otherwise, $\tilde{f}_\ell(x)$ is concentrated near lower membership degrees (e.g., $[0, 0.2]$).

Here, each $\tilde{f}_\ell(x)$ is a *set* of membership degrees rather than a single number, accommodating local variations. Conceptually, this divides $\mathbb{X}$ into overlapping "regions," each associated with a hyperfuzzy membership structure around x^ℓ.

Step 3: Defining Hyperfuzzy Rules. Let $\tilde{R} = \{\tilde{r}_1, \ldots, \tilde{r}_L\}$, where each rule $\tilde{r}_\ell$ is triggered mostly around x^ℓ. We denote an output function $y^\ell \approx f(x^\ell)$ for each ℓ. Then:

$$\tilde{r}_\ell : \quad \text{IF } \bigwedge_{i=1}^{n} \big(x_i \,\text{is}\, \tilde{\mu}_{\ell i}\big) \quad \text{THEN} \quad y \,\text{is}\, \tilde{\nu}_\ell,$$

where $\tilde{\mu}_{\ell i}$ is essentially $\tilde{f}_\ell$ focusing on the i-th dimension, and $\tilde{\nu}_\ell$ is a hyperfuzzy set capturing y^ℓ in $[0, 1]$ for some normalized representation or membership-coded output. The combination of these rules covers the entire input domain.

Step 4: Rule Aggregation and Defuzzification. The system aggregates contributions from each $\tilde{r}_\ell$. When x is near x^ℓ, $\tilde{f}_\ell(x)$ will be high (subset near 1). Thus, rule $\tilde{r}_\ell$ dominates the output, yielding an approximation close to $y^\ell \approx f(x^\ell)$. By uniform continuity, if x is near x^ℓ, $f(x)$ is near $f(x^\ell)$, ensuring

$$\|\widehat{f}(x) - f(x)\| < \varepsilon$$

for appropriate choice of membership boundaries and defuzzification methods.

Step 5: Learning Algorithm. The learning algorithm $\mathcal{L}_H$ refines these memberships and rules to further reduce approximation error. With gradient-based or other optimization methods, the membership sets $\tilde{f}_\ell$ and output sets $\tilde{\nu}_\ell$ converge to a configuration that keeps $\widehat{f}(x)$ within ε of $f(x)$.

Since each step can be made arbitrarily precise by refining the covering and adjusting membership sets, the Neuro-Hyperfuzzy System attains universal approximation. □

Theorem 4.8 (Universal Approximation for Neuro-n-SuperHyperfuzzy Systems)**.** *Let $\mathbb{X} \subseteq \mathbb{R}^n$ be compact, and let $f : \mathbb{X} \to \mathbb{R}^m$ be continuous. Then, for every $\varepsilon > 0$, there exists a Neuro-n-SuperHyperfuzzy System*

$$\mathcal{N}_{SH,n} = \left(\mathbb{X}, \mathbb{Y}, \tilde{F}_n, \tilde{R}_n, \mathcal{L}_{H,n}\right)$$

such that

$$\|\widehat{f}_n(x) - f(x)\| < \varepsilon \quad \textit{for all } x \in \mathbb{X},$$

where $\widehat{f}_n(x)$ is the system output under n-SuperHyperfuzzy membership.

Proof. The construction is similar to that in Theorem 4.7, but each membership function $\tilde{f}_{n,k}$ and rule $\tilde{r}_{n,k}$ uses an n-SuperHyperfuzzy Set, providing nested or higher-level uncertainties.

Step 1: Nested Membership Definition. For each grid point x^ℓ, define

$$\tilde{f}_{n,\ell} : \mathbb{X} \to \tilde{\mathcal{P}}_n([0,1]),$$

so that $\tilde{f}_{n,\ell}(x)$ includes multi-level membership subsets. Each level (1 through n) refines the degree of confidence or uncertainty, allowing flexible overlaps among different "regions" of x-space.

Step 2: n-SuperHyperfuzzy Rules. Construct q rules $\tilde{r}_{n,1}, \ldots, \tilde{r}_{n,q}$, each referencing $\tilde{f}_{n,\ell}$ membership sets. A typical rule might be:

$$\tilde{r}_{n,\ell} : \text{IF } \bigwedge_{i=1}^{n} \left(x_i \text{ is } \tilde{\mu}_{n,\ell i}\right) \quad \text{THEN} \quad y \text{ is } \tilde{\nu}_{n,\ell},$$

where $\tilde{\mu}_{n,\ell i}$ is a membership set in $\tilde{\mathcal{P}}_n([0,1])$ capturing the n-level membership for x_i, and $\tilde{\nu}_{n,\ell}$ similarly captures the output membership.

Step 3: Hierarchical Aggregation. When x is close to x^ℓ, the nested membership $\tilde{f}_{n,\ell}(x)$ becomes strongly activated at various levels, thus $\tilde{r}_{n,\ell}$ dominates the final output. Because we still rely on continuity of f and the refinement principle, we can ensure $\|\widehat{f}_n(x) - f(x)\| < \varepsilon$ by sufficiently dense covering in $\mathbb{X}$ and careful arrangement of nested membership sets.

Step 4: Learning Algorithm for n Levels. The learning process $\mathcal{L}_{H,n}$ can simultaneously tune membership sets at each hierarchy (from level 1 to level n). This does not limit approximation capacity; it merely provides additional structure for capturing uncertain or contradictory sources of information. By adjusting these levels, the system can approximate f within ε.

Hence, the Neuro-n-SuperHyperfuzzy System achieves universal approximation under the same topological and continuity assumptions as in the hyperfuzzy case. □

Remark 4.9. The above theorems emphasize that extending fuzzy membership to hyperfuzzy or n-SuperHyperfuzzy structures does not reduce the ability to approximate continuous functions on compact domains; rather, it enlarges the representational space of uncertainty.

Corollary 4.10. *In both Theorem 4.7 and Theorem 4.8, if the system is allowed to increase the number of rules q and refine membership sets arbitrarily, the approximation error can be made arbitrarily small.*

Proof. This corollary follows directly from the proofs of Theorems 4.7 and 4.8, where the coverings can be made finer and membership sets can be tuned more precisely as $q \to \infty$ or as membership boundaries become sharper. □

4.2. *Hyperfuzzy control*

A control system is a framework managing and regulating processes or devices to achieve desired outputs by adjusting inputs(cf. [3, 24, 73, 104, 115, 228]). The Control System is extended using Fuzzy Sets. Its main feature includes operations such as Fuzzification and Defuzzification. The Fuzzy Control System has been extensively studied in various research fields [53, 124, 130, 161, 173, 178, 182, 184, 205, 275, 277, 298]. The definition is provided below.

Definition 4.11. (cf. [67, 67, 140, 195]) Let $X = \{x_1, x_2, \ldots, x_n\}$ be the universe of discourse for input variables and $Y = \{y_1, y_2, \ldots, y_m\}$ for output variables. A fuzzy control system consists of the following components:

(1) *Fuzzification*: A mapping $F_x : X \to [0, 1]$ that transforms crisp input x_i into a fuzzy set $\mu(x_i)$, where $\mu(x_i) \in [0, 1]$ is the membership degree of x_i.
(2) *Fuzzy Rule Base*: A set of linguistic rules R_k of the form:
$$R_k : \text{If } x_1 \text{ is } A_1^k \text{ and } x_2 \text{ is } A_2^k \text{ and } \ldots \text{ then } y \text{ is } B^k,$$
where A_i^k and B^k are fuzzy sets defined on X and Y, respectively.
(3) *Inference Mechanism*: A function $\Phi : \mathcal{F}(X) \to \mathcal{F}(Y)$ that applies fuzzy logical operators (e.g., Min-Max or Max-Product) to infer the fuzzy output based on the rule base.
(4) *Defuzzification*: A process that converts the fuzzy output $\Phi(\mu(X))$ into a crisp value y^* using a defuzzification method such as:
$$y^* = \frac{\int_{y \in Y} y \cdot \mu(y)\, dy}{\int_{y \in Y} \mu(y)\, dy},$$
where $\mu(y)$ is the membership degree of y in the output fuzzy set.

Definition 4.12 (Hyperfuzzy Control System)**.** Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be universes of discourse for inputs and outputs, respectively. A *hyperfuzzy control system* extends the classical fuzzy control system by replacing each fuzzy set with a hyperfuzzy set:

(1) *Hyperfuzzification*: A mapping $\tilde{F}_x : X \to \tilde{P}([0,1])$, so each crisp input x_i is mapped to a *non-empty subset* of $[0,1]$ instead of a single membership degree.

(2) *Hyperfuzzy Rule Base*: A set of hyperfuzzy rules $\tilde{R}_k$ of the form:

$$\tilde{R}_k : \quad \text{If } x_1 \text{ is } \tilde{A}_1^k \text{ and } \dots \text{ then } y \text{ is } \tilde{B}^k,$$

where each $\tilde{A}_i^k, \tilde{B}^k$ is a hyperfuzzy set on X or Y.

(3) *Hyperfuzzy Inference*: An operator $\tilde{\Phi} : \tilde{\mathcal{F}}(X) \to \tilde{\mathcal{F}}(Y)$ that aggregates hyperfuzzy sets. Various extended t-norms/t-conorms or set-based operations can be used to combine membership *sets* instead of single values.

(4) *Hyperfuzzy Defuzzification*: A method mapping the final hyperfuzzy output $\tilde{\Phi}(\tilde{\mu}(X))$ into a crisp value y^*, potentially by selecting representative degrees from each subset or using bounding strategies:

$$y^* \;=\; \mathrm{Defz}\Big(\tilde{\Phi}\big(\tilde{\mu}(X)\big)\Big),$$

where $\mathrm{Defz}(\cdot)$ is an extended defuzzification operator for hyperfuzzy sets.

Definition 4.13 (n-SuperHyperfuzzy Control System)**.** Let X and Y be universes for inputs and outputs, respectively. An *n-SuperHyperfuzzy Control System* is defined by extending the hyperfuzzy components (Definition 4.12) to n nested levels:

(1) *n-SuperHyperfuzzification*: A mapping

$$\tilde{F}_{x,n} : X \;\to\; \tilde{\mathcal{P}}_n([0,1]),$$

associating each input $x_i \in X$ with an n-superhyperfuzzy set of membership degrees in $[0,1]$.

(2) *n-SuperHyperfuzzy Rule Base*: A set of rules $\tilde{R}_{n,k}$ of the form:

$$\tilde{R}_{n,k} : \quad \text{If } x_1 \text{ is } \tilde{A}_{1,n}^k \text{ and } \dots \text{ then } y \text{ is } \tilde{B}_n^k,$$

where each $\tilde{A}_{i,n}^k$, $\tilde{B}_n^k$ is an n-superhyperfuzzy set on X or Y.

(3) *n-SuperHyperfuzzy Inference*: An operator $\tilde{\Phi}_n$ that merges the multi-level membership subsets according to extended logic operations.

(4) *n-SuperHyperfuzzy Defuzzification*: A process converting the final multi-level hyperfuzzy output into a crisp control output y^*.

Here we present two illustrative theorems on stability and robustness. They assume a dynamical system model:

$$\dot{x}(t) \;=\; F\big(x(t), u(t)\big),$$

where $x(t)$ is the system state and $u(t)$ is the control input produced by a hyperfuzzy or n-superhyperfuzzy controller.

Definition 4.14 (Robustness)**.** (cf. [163, 287]) Robustness refers to the ability of a system, model, or algorithm to maintain its performance or functionality under perturbations, uncertainties, or adverse conditions in its inputs or environment.

Theorem 4.15 (Stability under Hyperfuzzy Control)**.** *Consider a control system with state-space model $\dot{x}(t) = F\big(x(t), u_{\mathrm{HF}}(t)\big)$, where $u_{\mathrm{HF}}(t)$ is generated by a hyperfuzzy control system. Suppose:*

(1) *$F(\cdot,\cdot)$ is continuous and locally Lipschitz in x for each fixed u_{HF}.*

(2) *There exists a hyperfuzzy rule base ensuring that for each neighborhood of the equilibrium $x^* = 0$, the hyperfuzzy inference produces a control input u_{HF} that reduces a Lyapunov function $V(x)$ (cf. [27, 62, 266, 281]).*

Then $x = 0$ is Lyapunov-stable; i.e., for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \varepsilon$ for all $t > 0$.

Proof. Lyapunov Function Construction. By hypothesis, we have a scalar function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ continuous, with $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. Further, assume V is radially unbounded in a local neighborhood of interest.

Hyperfuzzy Control Influence. The hyperfuzzy rule base ensures that when x is near 0, the membership sets produce a control u_{HF} such that $\dot{V}(x) = \nabla V(x) \cdot F(x, u_{\mathrm{HF}})$ remains non-positive. Since membership sets in $[0, 1]$ are now replaced by $\subseteq [0, 1]$, the control u_{HF} can be chosen consistently from these subsets to maintain $\dot{V}(x) \leq 0$.

Local Stability Conclusion. By standard Lyapunov arguments, if $\dot{V}(x) \leq 0$ in a neighborhood around $x = 0$, $x = 0$ is stable. More precisely, for any $\varepsilon > 0$, choose a region $\Omega_\varepsilon = \{x : V(x) < \gamma(\varepsilon)\}$ that implies $\|x\| < \varepsilon$. The local Lipschitz continuity of F and continuity of V confirm that once $x(0)$ is in Ω_ε, $V(x(t))$ cannot increase, hence $x(t)$ remains in Ω_ε. Thus $x = 0$ is Lyapunov-stable. □

Theorem 4.16 (Robustness under n-SuperHyperfuzzy Control)**.** *Let $\dot{x}(t) = F\big(x(t), u_{n\text{-}SHF}(t)\big)$ be controlled by an n-superhyperfuzzy system (Definition 4.13) with multi-level membership sets. Suppose:*

(1) *$F(x, u)$ is continuous in (x, u) and locally Lipschitz in x for each feasible u.*

(2) *The n-superhyperfuzzy rule base can generate control inputs that counteract bounded external disturbances $d(t)$ up to a known magnitude $D_{\max}$.*

Then the closed-loop system is robustly stable against disturbances of magnitude at most $D_{\max}$, provided the n-superhyperfuzzy sets at each level are appropriately tuned to reduce a Lyapunov function or maintain an invariant set.

Proof. We introduce an augmented system model:

$$\dot{x}(t) = F\big(x(t), u_{n\text{-SHF}}(t)\big) + d(t),$$

with $\|d(t)\| \leq D_{\max}$. By the assumption on the n-superhyperfuzzy rule base, each membership set at level $k = 1, \ldots, n$ provides a "range" of possible control actions. For any x in a certain neighborhood, one can choose an action $u_{n\text{-SHF}}$ from these superhyperfuzzy sets to balance or compensate for $d(t)$ within $\|d(t)\| \leq D_{\max}$.

Let $V(x)$ be a Lyapunov function that decreases under $u_{n\text{-SHF}}$ in the absence of disturbance. Because the multi-level membership offers flexible or nested sets of control values, for every x with $\|x\| \leq R$, there exists $u_{n\text{-SHF}}(x)$ such that $\dot{V}(x) \leq -\alpha(\|x\|)$ if $\|d(t)\| \leq D_{\max}$, for some positive function α. Thus $V(x(t))$ cannot grow unbounded. Consequently, $x(t)$ remains in a bounded region (or converges near 0) even under external disturbance up to $D_{\max}$. This shows robust stability in the sense that the system can handle disturbances within the specified bound. □

Remark 4.17. In Theorem 4.16, the nested membership structure of n-superhyperfuzzy sets allows the control logic to switch or adapt among multiple levels of uncertainty, improving robustness compared to single-level hyperfuzzy or classical fuzzy approaches.

5. **Future Work: Further Exploration of HyperUncertain Extensions**

This section briefly outlines future directions for this research. We anticipate advancements in extending various concepts using Hyperfuzzy Sets, SuperHyperfuzzy Sets, HyperNeutrosophic Sets, SuperHyperNeutrosophic Sets, Hyperplithogenic Sets, and SuperHyperplithogenic Sets.

Potential areas of extension include:

- *Neutrosophic Queueing Systems* [57, 214, 294].
- *Neutrosophic Geometry* [5, 153, 157].
- *Fuzzy Queueing Systems* [152, 204].
- *Fuzzy Matroids* [108, 109, 112, 169, 191, 224].
- *Fuzzy Topology* [4, 171, 209, 263, 269, 293].
- *Fuzzy Geometry* [48, 197, 218, 219].

We hope this research inspires further exploration and expansion in these domains.

Funding

No external funding was received for this research.

Acknowledgments

We extend our heartfelt thanks to everyone who provided invaluable support in completing this paper; this work would not have been possible without the contributions of all those involved. We also express our deep respect and gratitude to the authors of prior studies cited in our references. Lastly, we are sincerely thankful to our readers for their time and interest.

Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[2] Ralph Abboud, Ismail Ilkan Ceylan, Martin Grohe, and Thomas Lukasiewicz. The surprising power of graph neural networks with random node initialization. *arXiv preprint arXiv:2010.01179*, 2020.

[3] Tarek F. Abdelzaher, Yixin Diao, Joseph L. Hellerstein, Chenyang Lu, and Xiaoyun Zhu. Introduction to control theory and its application to computing systems. 2008.

[4] Nasim Abdolmaleki, M. Ahmadi, Hadi Tabatabaee Malazi, and Sebastiano Milardo. Fuzzy topology discovery protocol for sdn-based wireless sensor networks. *Simul. Model. Pract. Theory*, 79:54–68, 2017.

[5] Mohammad Abobala and Ahmad Hatip. An algebraic approach to neutrosophic euclidean geometry. *Neutrosophic Sets and Systems*, 43(1):10, 2021.

[6] Ather Abdulrahman Ageeli. A neutrosophic decision-making methods of the key aspects for supply chain management in international business administrations. *International Journal of Neutrosophic Science*, 23(1):155–167, 2023.

[7] Muhammad Akram and Noura Omair Alshehri. Intuitionistic fuzzy cycles and intuitionistic fuzzy trees. *The Scientific World Journal*, 2014(1):305836, 2014.

[8] Qeethara Al-Shayea. Artificial neural networks in medical diagnosis. *International Journal of Research Publication and Reviews*, 2024.

[9] Salah Hasan Saleh Al-subhi, Iliana Pérez Pupo, Roberto García Vacacela, Pedro Piñero, Pérez, and Maikel Yelandi Leyva Vázquez. A new neutrosophic cognitive map with neutrosophic sets on connections, application in project management. *Neutrosophic Sets and Systems*, 22:63–75, 2018.

[10] Mumtaz Ali, Florentin Smarandache, and Mohsin Khan. Study on the development of neutrosophic triplet ring and neutrosophic triplet field. *Mathematics*, 6(4):46, 2018.

[11] Alireza Aliahmadi and Hamed Nozari. Evaluation of security metrics in aiot and blockchain-based supply chain by neutrosophic decision-making method. *Supply Chain Forum: An International Journal*, 24:31 – 42, 2022.

[12] Keith Allan. Classifiers. *Language*, 53(2):285–311, 1977.

[13] Uri Alon and Eran Yahav. On the bottleneck of graph neural networks and its practical implications. *arXiv preprint arXiv:2006.05205*, 2020.

[14] Andreas S Andreou, NH Mateou, and George A Zombanakis. Optimization in genetically evolved fuzzy cognitive maps supporting decision-making: the limit cycle case. In *Proceedings. 2004 International Conference on Information and Communication Technologies: From Theory to Applications, 2004.*, pages 377–378. IEEE, 2004.

[15] Abdul Quaiyum Ansari, Ranjit Biswas, and Swati Aggarwal. Neutrosophic classifier: An extension of fuzzy classifer. *Applied soft computing*, 13(1):563–573, 2013.

[16] Abdul Quaiyum Ansari, Priyanka Sharma, and Manjari Tripathi. Automatic seizure detection using neutrosophic classifier. *Physical and Engineering Sciences in Medicine*, 43:1019–1028, 2020.

[17] Ioannis D. Apostolopoulos and Peter P. Groumpos. Fuzzy cognitive maps: Their role in explainable artificial intelligence. *Applied Sciences*, 2023.

[18] Charles Ashbacher. *Introduction to Neutrosophic logic*. Infinite Study, 2014.

[19] Robert Axelrod. Structure of decision : the cognitive maps of political elites. 2015.

[20] Jimmy Lei Ba. Layer normalization. *arXiv preprint arXiv:1607.06450*, 2016.

[21] Francis Bach. Exploring large feature spaces with hierarchical multiple kernel learning. *Advances in neural information processing systems*, 21, 2008.

[22] Jørgen Bang-Jensen and Gregory Gutin. *Classes of directed graphs*, volume 11. Springer, 2018.

[23] Peter L Bartlett, Dylan J Foster, and Matus J Telgarsky. Spectrally-normalized margin bounds for neural networks. *Advances in neural information processing systems*, 30, 2017.

[24] Robert N. Bateson. Introduction to control system technology. 1973.

[25] Richard Bellman. *Introduction to matrix analysis*. SIAM, 1997.

[26] Anushree Bhattacharya and Madhumangal Pal. A fuzzy graph theory approach to the facility location problem: A case study in the indian banking system. *Mathematics*, 11(13):2992, 2023.

[27] Franco Blanchini. Nonquadratic lyapunov functions for robust control. *Autom.*, 31:451–461, 1995.

[28] Lashon B. Booker, David E. Goldberg, and John H. Holland. Classifier systems and genetic algorithms. *Artif. Intell.*, 40:235–282, 1989.

[29] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[30] Salah Bouzina. Fuzzy logic vs. neutrosophic logic: operations logic. *Neutrosophic Sets and Systems*, 14:29–34, 2016.

[31] Robert M Brooks and Klaus Schmitt. The contraction mapping principle and some applications. *Electronic Journal of Differential Equations*, pages 09–90, 2009.

[32] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[33] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[34] Richard Hubert Bruck. *A survey of binary systems*, volume 20. Springer, 1971.

[35] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, linda Qiao, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *International Joint Conference on Artificial Intelligence*, 2022.

[36] Matteo Carandini and David J Heeger. Normalization as a canonical neural computation. *Nature reviews neuroscience*, 13(1):51–62, 2012.

[37] Joao Paulo Carvalho. On the semantics and the use of fuzzy cognitive maps and dynamic cognitive maps in social sciences. *Fuzzy Sets Syst.*, 214:6–19, 2013.

[38] Y. V. M. Cepeda, M. A. R. Guevara, E. J. J. Mogro, and R. P. Tizano. Impact of irrigation water technification on seven directories of the san juan-patoa river using plithogenic n-superhypergraphs based on environmental indicators in the canton of pujili, 2021. *Neutrosophic Sets and Systems*, 74:46–56, 2024.

[39] Xiaoguang Chang and John H Lilly. Evolutionary design of a fuzzy classifier from data. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 34(4):1894–1906, 2004.

[40] Liang Chen, Paul Bentley, Kensaku Mori, Kazunari Misawa, Michitaka Fujiwara, and Daniel Rueckert. Self-supervised learning for medical image analysis using image context restoration. *Medical image analysis*, 58:101539, 2019.

[41] Wei Chen, Wei Chen, Hamid Reza Pourghasemi, M. Panahi, Aiding Kornejady, Jiale Wang, Xiaoshen Xie, and Shubo Cao. Spatial prediction of landslide susceptibility using an adaptive neuro-fuzzy inference system combined with frequency ratio, generalized additive model, and support vector machine techniques. *Geomorphology*, 297:69–85, 2017.

[42] Wei Chen, M. Panahi, Paraskevas Tsangaratos, Himan Shahabi, Ioanna Ilia, Somayeh Panahi, Shao jun Li, Abolfazl Jaafari, and Baharin Bin Ahmad. Applying population-based evolutionary algorithms and a neuro-fuzzy system for modeling landslide susceptibility. *CATENA*, 2019.

[43] Graciela Chichilnisky. Social aggregation rules and continuity. *The Quarterly Journal of Economics*, 97(2):337–352, 1982.

[44] Jungwook Choi, Zhuo Wang, Swagath Venkataramani, Pierce I-Jen Chuang, Vijayalakshmi Srinivasan, and Kailash Gopalakrishnan. Pact: Parameterized clipping activation for quantized neural networks. *arXiv preprint arXiv:1805.06085*, 2018.

[45] Victor Christianto and Florentin Smarandache. A review of seven applications of neutrosophic logic: In cultural psychology, economics theorizing, conflict resolution, philosophy of science, etc. *J*, 2018.

[46] Zhang Chunying, Liu Lu, Ouyang Dong, and Liang Ruitao. Research of rough cognitive map model. In *Advanced Research on Electronic Commerce, Web Application, and Communication: International Conference, ECWAC 2011, Guangzhou, China, April 16-17, 2011. Proceedings, Part II*, pages 224–229. Springer, 2011.

[47] Zhang Chunying, Liu Lu, Liang Ruitao, and Wang Jing. Rough center mining algorithm of rough cognitive map. *Procedia engineering*, 15:3461–3465, 2011.

[48] Terry D Clark, Jennifer M Larson, John N Mordeson, Joshua D Potter, Mark J Wierman, Terry D Clark, Jennifer M Larson, John N Mordeson, Joshua D Potter, and Mark J Wierman. Fuzzy geometry. *Applying Fuzzy Mathematics to Formal Models in Comparative Politics*, pages 65–80, 2008.

[49] Veysel Çoban and Sezi Çevik Onar. Strategic analysis of solar energy pricing process with hesitant fuzzy cognitive map. *Energy Management?Collective and Computational Intelligence with Theory and Applications*, pages 195–227, 2018.

[50] Veysel Çoban and Sezi Çevik Onar. Analysis of solar energy generation capacity using hesitant fuzzy cognitive maps. *International Journal of Computational Intelligence Systems*, 10(1):1149–1167, 2017.

[51] Veysel Çoban and Sezi Çevik Onar. Modeling renewable energy usage with hesitant fuzzy cognitive map. *Complex & Intelligent Systems*, 3:155–166, 2017.

[52] Mihaela Colhon, Monica Tilea, Ana González-Marcos, Alina Stela Resceanu, Florentin Smarandache, and Fermín Navaridas-Nalda. A neutrosophic decision-making model for determining young people's active engagement. *Int. J. Inf. Technol. Decis. Mak.*, 23:569–598, 2023.

[53] Mario Collotta, Renato Ferrero, Edoardo Giusto, Mohammad Ghazi Vakili, Jacopo Grecuccio, Xiangjie Kong, and Ilsun You. A fuzzy control system for energy-efficient wireless devices in the internet of vehicles. *International Journal of Intelligent Systems*, 36:1595 – 1618, 2021.

[54] Irving M Copi, Carl Cohen, and Kenneth McMahon. *Introduction to logic*. Routledge, 2016.

[55] Pádraig Cunningham, Matthieu Cord, and Sarah Jane Delany. Supervised learning. In *Machine learning techniques for multimedia: case studies on organization and retrieval*, pages 21–49. Springer, 2008.

[56] Theodoros Damoulas and Mark A Girolami. Combining feature spaces for classification. *Pattern Recognition*, 42(11):2671–2683, 2009.

[57] K Dayana, T Poongodi, and B Vennila. Study on single server finite capacity neutrosophic queueing model. *Computational and Applied Mathematics*, 44(1):1–30, 2025.

[58] Ailin Deng and Bryan Hooi. Graph neural network-based anomaly detection in multivariate time series. In *AAAI Conference on Artificial Intelligence*, 2021.

[59] Remi Denton, Sam Gross, and Rob Fergus. Semi-supervised learning with context-conditional generative adversarial networks. *arXiv preprint arXiv:1611.06430*, 2016.

[60] Muhammet Deveci, Vladimir Simić, and Ali Ebadi Torkayesh. Remanufacturing facility location for automotive lithium-ion batteries: An integrated neutrosophic decision-making model. *Journal of Cleaner Production*, 317:128438, 2021.

[61] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[62] Moritz Diehl, Rishi Amrit, and James B. Rawlings. A lyapunov function for economic optimizing model predictive control. *IEEE Transactions on Automatic Control*, 56:703–707, 2011.

[63] Joseph Diestel and B Faires. On vector measures. *Transactions of the American Mathematical Society*, 198:253–271, 1974.

[64] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[65] Reinhard Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173(33):12, 2005.

[66] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[67] Dimiter Driankov, Hans Hellendoorn, and Michael Reinfrank. An introduction to fuzzy control (2nd ed.). 1996.

[68] Shiv Ram Dubey, Satish Kumar Singh, and Bidyut Baran Chaudhuri. Activation functions in deep learning: A comprehensive survey and benchmark. *Neurocomputing*, 503:92–108, 2022.

[69] Mehtap Dursun and Guray Gumus. Intuitionistic fuzzy cognitive map approach for the evaluation of supply chain configuration criteria. *Mathematical Methods in the Applied Sciences*, 43(13):7788–7801, 2020.

[70] Philip Ehrlich. *Real numbers, generalizations of the reals, and theories of continua*, volume 242. Springer Science & Business Media, 2013.

[71] Mohamed Elhoseny, Mahmoud Abdel-salam, and Ibrahim M Elhasnony. Extended fuzzy neutrosophic classifier for accurate intrusion detection and classification. *International Journal of Neutrosophic Science (IJNS)*, 24(4), 2024.

[72] Herbert B. Enderton. A mathematical introduction to logic. 1972.

[73] Virgil Eveleigh and David P. Lindorff. Introduction to control system design. *IEEE Transactions on Systems, Man, and Cybernetics*, 3:529–529, 1973.

[74] Wenqi Fan, Yao Ma, Qing Li, Yuan He, Eric Zhao, Jiliang Tang, and Dawei Yin. Graph neural networks for social recommendation. In *The world wide web conference*, pages 417–426, 2019.

[75] Ilijas Farah. Approximate homomorphisms. *Combinatorica*, 18:335–348, 1998.

[76] Mehdi Fasanghari and Farzad Habibipour Roudsari. The fuzzy evaluation of e-commerce customer satisfaction. 2008.

[77] Noha S Fayed, Mohammed M Elmogy, Ahmed Atwan, and Eman El-Daydamony. Efficient occupancy detection system based on neutrosophic weighted sensors data fusion. *IEEE Access*, 10:13400–13427, 2022.

[78] Shuang Feng and C.L. Philip Chen. Fuzzy broad learning system: A novel neuro-fuzzy model for regression and classification. *IEEE Transactions on Cybernetics*, 50:414–424, 2020.

[79] Yifan Feng, Haoxuan You, Zizhao Zhang, R. Ji, and Yue Gao. Hypergraph neural networks. In *AAAI Conference on Artificial Intelligence*, 2018.

[80] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 437.

[81] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. 2024.

[82] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. Technical report, Center for Open Science, 2024.

[83] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[84] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[85] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[86] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[87] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[88] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[89] Takaaki Fujita. A theoretical exploration of hyperconcepts: Hyperfunctions, hyperrandomness, hyperdecision-making, and beyond (including a survey of hyperstructures). 2024.

[90] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[91] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[92] Takaaki Fujita. A theoretical exploration of hyperconcepts: Hyperfunctions, hyperrandomness, hyperdecision-making, and beyond (including a survey of hyperstructures). *Preprint in Researchgate*, 2025.

[93] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[94] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[95] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[96] Takaaki Fujita and Florentin Smarandache. Fundamental computational problems and algorithms for superhypergraphs. 2024.

[97] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[98] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[99] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[100] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[101] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[102] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[103] A Nagoor Gani and K Radha. On regular fuzzy graphs. 2008.

[104] K. C. Garner. Introduction to control system performance measurements. 1968.

[105] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[106] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[107] Puspendu Giri, Somnath Paul, and Bijoy Krishna Debnath. A fuzzy graph theory and matrix approach (fuzzy gtma) to select the best renewable energy alternative in india. *Applied Energy*, 358:122582, 2024.

[108] Roy Goetschel and William Voxman. Fuzzy matroids. 1988.

[109] Roy Goetschel and William Voxman. Fuzzy matroid sums and a greedy algorithm. *Fuzzy Sets and Systems*, 37:189–200, 1990.

[110] M Eshaghi Gordji, M Ramezani, Manuel De La Sen, and Yeol Je Cho. On orthogonal sets and banach fixed point theorem. *Fixed Point Theory*, 18(2):569–578, 2017.

[111] Andrzej Granas, James Dugundji, et al. *Fixed point theory*, volume 14. Springer, 2003.

[112] Fu gui Shi. (l, m)-fuzzy matroids. *Fuzzy Sets Syst.*, 160:2387–2400, 2009.

[113] Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):203, 2018.

[114] Krystal Guo and Bojan Mohar. Hermitian adjacency matrix of digraphs and mixed graphs. *Journal of Graph Theory*, 85(1):217–248, 2017.

[115] Francis Joseph Hale. Introduction to control system analysis and design. 1973.

[116] Mohammad Hamidi, Florentin Smarandache, and Mohadeseh Taghinezhad. *Decision Making Based on Valued Fuzzy Superhypergraphs*. Infinite Study, 2023.

[117] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[118] Barbara Hammer and Kai Gersmann. A note on the universal approximation capability of support vector machines. *neural processing letters*, 17:43–53, 2003.

[119] Trevor Hastie, Robert Tibshirani, Jerome Friedman, Trevor Hastie, Robert Tibshirani, and Jerome Friedman. Overview of supervised learning. *The elements of statistical learning: Data mining, inference, and prediction*, pages 9–41, 2009.

[120] Robert Hecht-Nielsen. Theory of the backpropagation neural network. In *Neural networks for perception*, pages 65–93. Elsevier, 1992.

[121] R. Hema, R. Sudharani, and M. Kavitha. A novel approach on plithogenic interval valued neutrosophic hypersoft sets and its application in decision making. *Indian Journal Of Science And Technology*, 2023.

[122] Nasimeh Heydaribeni, Xinrui Zhan, Ruisi Zhang, Tina Eliassi-Rad, and Farinaz Koushanfar. Hypop: Distributed constrained combinatorial optimization leveraging hypergraph neural networks. *ArXiv*, abs/2311.09375, 2023.

[123] Doeko Homan. Very basic set theory. 2023.

[124] Amirsoheil Honarbari, Sajad Najafi-Shad, Mohsen Saffari Pour, S. S. M. Ajarostaghi, and Ali Hassannia. Mppt improvement for pmsg-based wind turbines using extended kalman filter and fuzzy control system. *Energies*, 2021.

[125] Haoyuan Hong, M. Panahi, Ataollah Shirzadi, Tianwu Ma, Junzhi Liu, A-Xing Zhu, Wei Chen, Ioannis Kougias, and Nerantzis Kazakis. Flood susceptibility assessment in hengfeng area coupling adaptive neuro-fuzzy inference system with genetic algorithm and differential evolution. *The Science of the total environment*, 621:1124–1141, 2018.

[126] WA267432 Horn. Some fixed point theorems for compact maps and flows in banach spaces. *Transactions of the American Mathematical Society*, 149(2):391–404, 1970.

[127] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(1):35–40, 1990.

[128] Chao Hu, Ruishi Yu, Binqi Zeng, Yu Zhan, Ying Fu, Quan Zhang, Rongkai Liu, and Heyuan Shi. Hyperattack: Multi-gradient-guided white-box adversarial structure attack of hypergraph neural networks. *ArXiv*, abs/2302.12407, 2023.

[129] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[130] Jiage Huo, Jianghua Zhang, and Felix T. S. Chan. A fuzzy control system for assembly line balancing with a three-state degradation process in the era of industry 4.0. *International Journal of Production Research*, 58:7112 – 7129, 2020.

[131] S Satham Hussain, N Durga, Rahmonlou Hossein, and Ghorai Ganesh. New concepts on quadripartitioned single-valued neutrosophic graph with real-life application. *International Journal of Fuzzy Systems*, 24(3):1515–1529, 2022.

[132] S Satham Hussain, Hossein Rashmonlou, R Jahir Hussain, Sankar Sahoo, Said Broumi, et al. Quadripartitioned neutrosophic graph structures. *Neutrosophic Sets and Systems*, 51(1):17, 2022.

[133] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[134] Dimitris K Iakovidis and Elpiniki Papageorgiou. Intuitionistic fuzzy cognitive maps for medical decision making. *IEEE Transactions on Information Technology in Biomedicine*, 15(1):100–107, 2010.

[135] Aulia Ishak, R. Ginting, and Wang Wanli. Evaluation of e-commerce services quality using fuzzy ahp and topsis. *IOP Conference Series: Materials Science and Engineering*, 1041, 2021.

[136] Abolfazl Jaafari, Eric K. Zenner, M. Panahi, and Himan Shahabi. Hybrid artificial intelligence models based on a neuro-fuzzy system and metaheuristic optimization algorithms for spatial prediction of wildfire probability. *Agricultural and Forest Meteorology*, 2019.

[137] Muhammad Naveed Jafar, Muhammad Haris Saeed, Kainat Muniba Khan, Faten S. Alamri, and Hamiden A. Wahed Khalifa. Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. *IEEE Access*, PP:1–1, 2022.

[138] Chiranjibe Jana, Tapan Senapati, Monoranjan Bhowmik, and Madhumangal Pal. On intuitionistic fuzzy g-subalgebras of g-algebras. *Fuzzy Information and Engineering*, 7(2):195–209, 2015.

[139] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[140] David F. Jenkins and Kevin M. Passino. An introduction to nonlinear analysis of fuzzy control systems. *J. Intell. Fuzzy Syst.*, 7:75–103, 1999.

[141] Jianwen Jiang, Yuxuan Wei, Yifan Feng, Jingxuan Cao, and Yue Gao. Dynamic hypergraph neural networks. In *International Joint Conference on Artificial Intelligence*, 2019.

[142] Weiwei Jiang and Jiayun Luo. Graph neural network for traffic forecasting: A survey. *ArXiv*, abs/2101.11174, 2021.

[143] Calvin Jongsma. Basic set theory and combinatorics. *Undergraduate Texts in Mathematics*, 2019.

[144] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[145] Young Bae Jun, Min Su Kang, and Seok Zun Song. Several types of bipolar fuzzy hyper bck-ideals in hyper bck-algebras. 2012.

[146] Young Bae Jun, Seon Jeong Kim, and Seok Zun Song. Hyper permeable values and energetic sets in bck/bci-algebras. 2020.

[147] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[148] Dr.W.B.Vasantha Kandasamy and Florentin Smarandache. Fuzzy cognitive maps and neutrosophic cognitive maps. 2003.

[149] Ilanthenral Kandasamy and Florentin Smarandache. Algebraic structure of neutrosophic duplets in neutrosophic rings. 2018.

[150] Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Neutrosophic graphs: a new dimension to graph theory*. Infinite Study, 2015.

[151] Bingyi Kang, Yu Li, Sa Xie, Zehuan Yuan, and Jiashi Feng. Exploring balanced feature spaces for representation learning. In *International conference on learning representations*, 2020.

[152] Chiang Kao, Chang-Chung Li, and Shih-Pin Chen. Parametric programming to the analysis of fuzzy queues. *Fuzzy sets and systems*, 107(1):93–100, 1999.

[153] Chaitali Kar, Bappa Mondal, and Tapan Kumar Roy. An inventory model under space constraint in neutrosophic environment: a neutrosophic geometric programming approach. *Neutrosophic Sets and Systems: An International Book Series in Information Science and Engineering*, 21(2018):93–109, 2018.

[154] B Kavitha, S Karthikeyan, and P Sheeba Maybell. Emerging intuitionistic fuzzy classifiers for intrusion detection system. *Journal of Advances in Information Technology*, 2(2):99–108, 2011.

[155] M Kaviyarasu, Muhammad Aslam, Farkhanda Afzal, Maha Mohammed Saeed, Arif Mehmood, and Saeed Gul. The connectivity indices concept of neutrosophic graph and their application of computer network, highway system and transport network flow. *Scientific Reports*, 14(1):4891, 2024.

[156] Faezehossadat Khademi, Sayed Mohammadmehdi Jamal, Neela Deshpande, and Shreenivas N. Londhe. Predicting strength of recycled aggregate concrete using artificial neural network, adaptive neuro-fuzzy inference system and multiple linear regression. *International journal of sustainable built environment*, 5:355–369, 2016.

[157] Huda E Khalid, Florentin Smarandache, and Ahmed K Essa. *The basic notions for (over, off, under) neutrosophic geometric programming problems*. Infinite Study, 2018.

[158] Ali Khosravi, Ricardo Nicolau Nassar Koury, Luiz Henrique Jorge Machado, and Juan J. Garcia Pabon. Prediction of wind speed and wind direction using artificial neural network, support vector regression and adaptive neuro-fuzzy inference system. *Sustainable Energy Technologies and Assessments*, 25:146–160, 2018.

[159] Günter Klambauer, Thomas Unterthiner, Andreas Mayr, and Sepp Hochreiter. Self-normalizing neural networks. *Advances in neural information processing systems*, 30, 2017.

[160] Stephen Cole Kleene. General recursive functions of natural numbers. *Mathematische annalen*, 112(1):727–742, 1936.

[161] Dimitrios Kontogiannis, Dimitrios Bargiotas, and Aspassia Daskalopulu. Fuzzy control system for smart energy management in residential buildings based on environmental data. *Energies*, 2021.

[162] Bart Kosko. Fuzzy cognitive maps. *Int. J. Man Mach. Stud.*, 24:65–75, 1986.

[163] Smita Krishnaswamy, Igor L. Markov, and John P. Hayes. Design for robustness. 2013.

[164] Tarun Kumar and Mukesh Kumar Sharma. Neutrosophic decision-making for allocations in solid transportation problems. *OPSEARCH*, 2024.

[165] Ludmila I Kuncheva. *Fuzzy classifier design*, volume 49. Physica, 2012.

[166] Shulin Lan, Hao Zhang, Ray Y. Zhong, and George Q. Huang. A customer satisfaction evaluation model for logistics services using fuzzy analytic hierarchy process. *Ind. Manag. Data Syst.*, 116:1024–1042, 2016.

[167] Jérôme Lang, Gabriella Pigozzi, Marija Slavkovik, and Leendert Van der Torre. Judgment aggregation rules based on minimization. In *Proceedings of the 13th conference on theoretical aspects of rationality and knowledge*, pages 238–246, 2011.

[168] Maikel Leon. Aggregating procedure for fuzzy cognitive maps. *The International FLAIRS Conference Proceedings*, 2023.

[169] Xiaonan Li. Three-way fuzzy matroids and granular computing. *International Journal of Approximate Reasoning*, 114:44–50, 2019.

[170] Timothy P Lillicrap, Adam Santoro, Luke Marris, Colin J Akerman, and Geoffrey Hinton. Backpropagation and the brain. *Nature Reviews Neuroscience*, 21(6):335–346, 2020.

[171] Kimfung Liu, Wenzhong Shi, and Hua Zhang. A fuzzy topology-based maximum likelihood classification. *Isprs Journal of Photogrammetry and Remote Sensing*, 66:103–114, 2011.

[172] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[173] Zhi Liu, Ardashir Mohammadzadeh, Hamza Turabieh, Majdi M. Mafarja, Shahab S. Band, and Amir H. Mosavi. A new online learned interval type-3 fuzzy control system for solar energy management systems. *IEEE Access*, 9:10498–10508, 2021.

[174] Yingqi Lu, Pai Zhu, Donglin Wang, and Michel Fattouche. Machine learning techniques with probability vector for cooperative spectrum sensing in cognitive radio networks. In *2016 IEEE wireless communications and networking conference*, pages 1–6. IEEE, 2016.

[175] Chao Luo, Nannan Zhang, and Xingyuan Wang. Time series prediction based on intuitionistic fuzzy cognitive map. *Soft Computing*, 24:6835–6850, 2020.

[176] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.

[177] Mourad Oqla Massa'deh. On hyper q-fuzzy normal hx subgroup, conjugate and its.

[178] Willians Ribeiro Mendes, Fábio Meneghetti Ugulino de Araújo, Ritaban Dutta, and Derek M. Heeren. Fuzzy control system for variable rate irrigation using remote sensing. *Expert Syst. Appl.*, 124:13–24, 2019.

[179] Hamiyet Merkepci and Katy D. Ahmad. On the conditions of imperfect neutrosophic duplets and imperfect neutrosophic triplets. *Galoitica: Journal of Mathematical Structures and Applications*, 2022.

[180] Yuan Miao, Zhi qiang Liu, Shi Li, and Chee Kheong Siew. Dynamical cognitive network-an extension of fuzzy cognitive map. *Proceedings 11th International Conference on Tools with Artificial Intelligence*, pages 43–46, 1999.

[181] Arunodaya Raj Mishra, Dragan Pamucar, Pratibha Rani, Rajeev Shrivastava, and Ibrahim M. Hezam. Assessing the sustainable energy storage technologies using single-valued neutrosophic decision-making framework with divergence measure. *Expert Syst. Appl.*, 238:121791, 2023.

[182] Bijan Moaveni, Fatemeh Rashidi Fathabadi, and Ali Molavi. Fuzzy control system design for wheel slip prevention and tracking of desired speed profile in electric trains. *Asian Journal of Control*, 24:388 – 400, 2020.

[183] E. J. Mogro, J. R. Molina, G. J. S. Canas, and P. H. Soria. Tree tobacco extract (*Nicotiana glauca*) as a plithogenic bioinsecticide alternative for controlling fruit fly (*Drosophila immigrans*) using n-superhypergraphs. *Neutrosophic Sets and Systems*, 74:57–65, 2024.

[184] Ghazal Mohsenian, Sadegh Khalili, Mohammad I. Tradat, Yaman M. Manaserh, Srikanth Rangarajan, Anuroop Desu, Dushyant Thakur, Kourosh Nemati, Kanad Ghose, and Bahgat G. Sammakia. A novel integrated fuzzy control system toward automated local airflow management in data centers. *Control Engineering Practice*, 2021.

[185] Kalyan Mondal, Surapati Pramanik, and Nandalal Ghosh. A study on problems of hijras in west bengal based on neutrosophic cognitive maps. 2014.

[186] John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.

[187] Christopher Morris, Martin Ritzert, Matthias Fey, William L Hamilton, Jan Eric Lenssen, Gaurav Rattan, and Martin Grohe. Weisfeiler and leman go neural: Higher-order graph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 4602–4609, 2019.

[188] Giuseppe Munda. Choosing aggregation rules for composite indicators. *Social indicators research*, 109(3):337–354, 2012.

[189] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[190] Zohreh Nazari and Batool Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16:173 – 185, 2018.

[191] Ladislav A. Novak. On goetschel and voxman fuzzy matroids. *Fuzzy Sets Syst.*, 117:407–412, 2001.

[192] Sandra Oltra and Oscar Valero. Banach's fixed point theorem for partial metric spaces. 2004.

[193] Oluseyi Olurotimi, Amir Dembo, and Thomas Kailath. Neural network weight matrix synthesis using optimal control techniques. In *Neural Information Processing Systems*, 1989.

[194] Rodolfo González Ortega, Marcos David Oviedo Rodríguez, Maikel Yelandi Leyva Vázquez, Jesús Estupiñán, Ricardo, João Alcione Sganderla Figueiredo, and Florentin Smarandache. Pestel analysis based on neutrosophic cognitive maps and neutrosophic numbers for the sinos river basin management. 2019.

[195] Lena Osterhagen. An introduction to fuzzy control. 2016.

[196] Madhumangal Pal, Sovan Samanta, and Ganesh Ghorai. *Modern trends in fuzzy graph theory*. Springer, 2020.

[197] Sankar K. Pal and Ashish Ghosh. Fuzzy geometry in image analysis. *Fuzzy Sets and Systems*, 48:23–40, 1992.

[198] Dragan Pamucar, Morteza Yazdani, Radojko Obradovic, Anil Kumar, and Mercedes Torres-Jiménez. A novel fuzzy hybrid neutrosophic decision-making approach for the resilient supplier selection problem. *International Journal of Intelligent Systems*, 35(12):1934–1986, 2020.

[199] Elpiniki I. Papageorgiou. Review study on fuzzy cognitive maps and their applications during the last decade. *2011 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2011)*, pages 828–835, 2011.

[200] Elpiniki I Papageorgiou and Dimitris K Iakovidis. Intuitionistic fuzzy cognitive maps. *IEEE Transactions on Fuzzy Systems*, 21(2):342–354, 2012.

[201] Elpiniki I. Papageorgiou and Jose L. Salmeron. A review of fuzzy cognitive maps research during the last decade. *IEEE Transactions on Fuzzy Systems*, 21:66–79, 2013.

[202] Jooyoung Park and Irwin W Sandberg. Universal approximation using radial-basis-function networks. *Neural computation*, 3(2):246–257, 1991.

[203] Michael Peer, Iva K. Brunec, Nora S. Newcombe, and Russell A. Epstein. Structuring knowledge with cognitive maps and cognitive graphs. *Trends in Cognitive Sciences*, 25:37–54, 2020.

[204] Yannis A Phillis and Runtong Zhang. Fuzzy service control of queueing systems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 29(4):503–517, 1999.

[205] Robert Piotrowski. Supervisory fuzzy control system for biological processes in sequencing wastewater batch reactor. *Urban Water Journal*, 17:325 – 332, 2020.

[206] Tomaso Poggio and Federico Girosi. Networks for approximation and learning. *Proceedings of the IEEE*, 78(9):1481–1497, 1990.

[207] Surapati Pramanik, Sourendranath Chackrabarti, Nandalal Ghosh, and Kumar Ashutosh. A study on problems of construction workers in west bengal based on neutrosophic cognitive maps. *International Journal of Innovative Research in Science, Engineering and Technology*, 2:6387–6394, 2014.

[208] Mahardhika Pratama, Witold Pedrycz, and Edwin Lughofer. Evolving ensemble fuzzy classifier. *IEEE Transactions on Fuzzy Systems*, 26(5):2552–2567, 2018.

[209] Pao-Ming Pu and Ying ming Liu. Fuzzy topology. i. neighborhood structure of a fuzzy point and moore-smith convergence. *Journal of Mathematical Analysis and Applications*, 76:571–599, 1980.

[210] Dunwang Qin, Zhen Peng, and Lifeng Wu. Deep attention fuzzy cognitive maps for interpretable multivariate time series prediction. *Knowl. Based Syst.*, 275:110700, 2023.

[211] Jiezhong Qiu, Qibin Chen, Yuxiao Dong, Jing Zhang, Hongxia Yang, Ming Ding, Kuansan Wang, and Jie Tang. Gcc: Graph contrastive coding for graph neural network pre-training. *Proceedings of the 26th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, 2020.

[212] Mohammad Naved Qureshi and Mohd Vasim Ahamad. An improved method for image segmentation using k-means clustering with neutrosophic logic. *Procedia computer science*, 132:534–540, 2018.

[213] Atiqe Ur Rahman, Muhammad Haris Saeed, and Hamiden Abd El-Wahed Khalifa. Multi-attribute decision-making based on aggregations and similarity measures of neutrosophic hypersoft sets with possibility setting. *Journal of Experimental & Theoretical Artificial Intelligence*, 36:161 – 186, 2022.

[214] Heba Rashad and Mai Mohamed. Neutrosophic theory and its application in various queueing models: case studies. *Neutrosophic Sets and Systems*, 42:117–135, 2021.

[215] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[216] Raul Rojas and Raúl Rojas. The backpropagation algorithm. *Neural networks: a systematic introduction*, pages 149–182, 1996.

[217] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[218] Azriel Rosenfeld. The fuzzy geometry of image subsets. In *Readings in Fuzzy Sets for Intelligent Systems*, pages 633–639. Elsevier, 1993.

[219] Azriel Rosenfeld. Fuzzy geometry: An updated overview. *Information Sciences*, 110(3-4):127–133, 1998.

[220] Benjamin Rossman. Homomorphism preservation theorems. *J. ACM*, 55:15:1–15:53, 2008.

[221] Memet Şahin and Abdullah Kargın. *Neutrosophic triplet metric topology.* Infinite Study, 2019.

[222] Memet Şahin, Abdullah Kargın, and İsmet Yıldız. Neutrosophic triplet field and neutrosophic triplet vector space based on set valued neutrosophic quadruple number. *Quadruple Neutrosophic Theory And Applications*, 1:52, 2020.

[223] Alexei Samsonovich and Bruce L. McNaughton. Path integration and cognitive mapping in a continuous attractor neural network model. *The Journal of Neuroscience*, 17:5900 – 5920, 1997.

[224] Musavarah Sarwar and Muhammad Akram. New applications of m-polar fuzzy matroids. *Symmetry*, 9:319, 2017.

[225] S Satham Hussain, Durga Nagarajan, Hossein Rashmanlou, and Farshid Mofidnakhaei. Novel supply chain decision making model under m-polar quadripartitioned neutrosophic environment. *Journal of Applied Mathematics and Computing*, pages 1–26, 2024.

[226] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.

[227] Franco Scarselli, Marco Gori, Ah Chung Tsoi, Markus Hagenbuchner, and Gabriele Monfardini. The graph neural network model. *IEEE transactions on neural networks*, 20(1):61–80, 2008.

[228] Klaus Schmidt. Control system design. 2005.

[229] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[230] Sagar Sharma, Simone Sharma, and Anidhya Athaiya. Activation functions in neural networks. *Towards Data Sci*, 6(12):310–316, 2017.

[231] Oleksandr Shchur, Maximilian Mumme, Aleksandar Bojchevski, and Stephan Günnemann. Pitfalls of graph neural network evaluation. *ArXiv*, abs/1811.05868, 2018.

[232] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[233] Sajjan G. Shiva. Introduction to logic design. 2018.

[234] Tobias Sing, Oliver Sander, Niko Beerenwinkel, and Thomas Lengauer. Rocr: visualizing classifier performance in r. *Bioinformatics*, 21 20:3940–1, 2005.

[235] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[236] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[237] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[238] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[239] Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.

[240] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[241] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[242] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[243] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[244] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[245] Florentin Smarandache. Extended nonstandard neutrosophic logic, set, and probability based on extended nonstandard analysis. *Symmetry*, 11(4):515, 2019.

[246] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[247] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[248] Florentin Smarandache. *NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)*, volume 5. Infinite Study, 2021.

[249] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[250] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[251] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[252] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[253] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[254] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[255] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[256] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[257] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group. *Neural Computing and Applications*, 29(7):595–601, 2018.

[258] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[259] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[260] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[261] Florentin Smarandache and AA Salama. Neutrosophic crisp set theory. 2015.

[262] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[263] Alexander P Šostak. On a fuzzy topological structure. In *Proceedings of the 13th Winter School on Abstract Analysis*, pages 89–103. Circolo Matematico di Palermo, 1985.

[264] Teresa E Steele and Timothy D Weaver. The modified triangular graph: a refined method for comparing mortality profiles in archaeological samples. *Journal of Archaeological Science*, 29(3):317–322, 2002.

[265] Eulalia Szmidt, Janusz Kacprzyk, and Marta Kukier. Intuitionistic fuzzy classifier for imbalanced classes. In *Artificial Intelligence and Soft Computing: 12th International Conference, ICAISC 2013, Zakopane, Poland, June 9-13, 2013, Proceedings, Part I 12*, pages 483–492. Springer, 2013.

[266] Kazuo Tanaka, Tsuyoshi Hori, and Hua O. Wang. A multiple lyapunov function approach to stabilization of fuzzy control systems. *IEEE Trans. Fuzzy Syst.*, 11:582–589, 2003.

[267] Lev Telyatnikov, Maria Sofia Bucarelli, Guillermo Bernardez, Olga Zaghen, Simone Scardapane, and Pietro Lió. Hypergraph neural networks through the lens of message passing: A common perspective to homophily and architecture design. *ArXiv*, abs/2310.07684, 2023.

[268] Ankit Thakkar and Kinjal Chaudhari. Predicting stock trend using an integrated term frequency-inverse document frequency-based feature weight matrix with neural networks. *Appl. Soft Comput.*, 96:106684, 2020.

[269] S. P. Tiwari and Anupam K. Singh. Fuzzy preorder, fuzzy topology and fuzzy transition system. In *Indian Conference on Logic and Its Applications*, 2013.

[270] Vakkas Ulucay and Memet Sahin. Intuitionistic fuzzy soft expert graphs with application. *Uncertainty discourse and applications*, 1(1):1–10, 2024.

[271] TS Umamaheswari and P Sumathi. Enhanced firefly algorithm (efa) based gene selection and adaptive neuro neutrosophic inference system (annis) prediction model for detection of circulating tumor cells (ctcs) in breast cancer analysis. *Cluster Computing*, 22:14035–14047, 2019.

[272] Guillaume Verdon, Trevor McCourt, Enxhell Luzhnica, Vikash Singh, Stefan Leichenauer, and Jack Hidary. Quantum graph neural networks. *arXiv preprint arXiv:1909.12264*, 2019.

[273] MR Vinodh and PJ Arui Leena Rose. Intelligent deep adaptive intuitionistic fuzzy classifier based dyslexia prediction among children at its early stage. *Journal of Electrical Systems*, 20(3s):52–61, 2024.

[274] Jia Wang and Zhenyuan Wang. Using neural networks to determine sugeno measures by statistics. *Neural Networks*, 10:183–195, 1997.

[275] Lina Wang, Binrui Wang, and M. Zhu. Multi-model adaptive fuzzy control system based on switch mechanism in a greenhouse. *Applied Engineering in Agriculture*, 36:549–556, 2020.

[276] Yuxin Wang, Quan Gan, Xipeng Qiu, Xuanjing Huang, and David Paul Wipf. From hypergraph energy functions to hypergraph neural networks. In *International Conference on Machine Learning*, 2023.

[277] Zhanshan Wang, Jian Sun, and Huaguang Zhang. Stability analysis of t-s fuzzy control system with sampled-dropouts based on time-varying lyapunov function method. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50:2566–2577, 2020.

[278] Tong Wei, Junlin Hou, and Rui Feng. Fuzzy graph neural network for few-shot learning. In *2020 International joint conference on neural networks (IJCNN)*, pages 1–8. IEEE, 2020.

[279] Paul J Werbos. Backpropagation through time: what it does and how to do it. *Proceedings of the IEEE*, 78(10):1550–1560, 1990.

[280] Andrew M. Wikenheiser and Geoffrey Schoenbaum. Over the river, through the woods: cognitive maps in the hippocampus and orbitofrontal cortex. *Nature Reviews Neuroscience*, 17:513–523, 2016.

[281] F. Wesley Wilson. The structure of the level surfaces of a lyapunov function. *Journal of Differential Equations*, 3:323–329, 1967.

[282] Jianxin Wu. Introduction to convolutional neural networks. *National Key Lab for Novel Software Technology. Nanjing University. China*, 5(23):495, 2017.

[283] Zonghan Wu, Shirui Pan, Fengwen Chen, Guodong Long, Chengqi Zhang, and S Yu Philip. A comprehensive survey on graph neural networks. *IEEE transactions on neural networks and learning systems*, 32(1):4–24, 2020.

[284] Keyulu Xu, Weihua Hu, Jure Leskovec, and Stefanie Jegelka. How powerful are graph neural networks? *arXiv preprint arXiv:1810.00826*, 2018.

[285] Ibrahim Yasser, Abeer Twakol, AA Abd El-Khalek, Ahmed Samrah, and AA Salama. Covid-x: novel health-fog framework based on neutrosophic classifier for confrontation covid-19. *Neutrosophic Sets and Systems*, 35:1–21, 2020.

[286] Morteza Yazdani, Ali Ebadi Torkayesh, ?eljko Stević, Prasenjit Chatterjee, Sahand Asgharieh Ahari, and Violeta Doval Hernandez. An interval valued neutrosophic decision-making structure for sustainable supplier selection. *Expert Syst. Appl.*, 183:115354, 2021.

[287] Jyh-Cheng Yu and Kosuke Ishii. Design for robustness based on manufacturing variation patterns. *Journal of Mechanical Design*, 120:196–202, 1998.

[288] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[289] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[290] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[291] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[292] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[293] Lotfi A. Zadeh. Fuzzy topology. i. neighborhood structure of a fuzzy point and moore-smith convergence. 2003.

[294] Mohamed Bisher Zeina. Neutrosophic event-based queueing model. *International Journal of Neutrosophic Science*, 6(1):48–55, 2020.

[295] Chuxu Zhang, Dongjin Song, Chao Huang, Ananthram Swami, and N. Chawla. Heterogeneous graph neural network. *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, 2019.

[296] Xiaohong Zhang, Florentin Smarandache, and Xingliang Liang. Neutrosophic duplet semi-group and cancellable neutrosophic triplet groups. *Symmetry*, 9:275, 2017.

[297] Hua Zhao, Zeshui Xu, Shousheng Liu, and Zhong Wang. Intuitionistic fuzzy mst clustering algorithms. *Computers & Industrial Engineering*, 62(4):1130–1140, 2012.

[298] Liang Zhao, Shaocheng Qu, W. Zhang, and Zhili Xiong. An energy-saving fuzzy control system for highway tunnel lighting. *Optik*, 2019.

[299] Wufan Zhao, Claudio Persello, and Alfred Stein. Extracting planar roof structures from very high resolution images using graph neural networks. *ISPRS Journal of Photogrammetry and Remote Sensing*, 187:34–45, 2022.

[300] Ziwei Zheng, Huizhi Liang, Vaclav Snasel, Vito Latora, Panos Pardalos, Giuseppe Nicosia, and Varun Ojha. On learnable parameters of optimal and suboptimal deep learning models. *arXiv preprint arXiv:2408.11720*, 2024.

[301] Enwang Zhou and Alireza Khotanzad. Fuzzy classifier design using genetic algorithms. *Pattern Recognition*, 40(12):3401–3414, 2007.

Chapter 3

Some Types of HyperNeutrosophic Set: Bipolar, Pythagorean, Double-Valued, Interval-Valued Set

Takaaki Fujita [1] * *and Florentin Smarandache* [2]
[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.; t171d603@gunma-u.ac.jp.
[2] Department of Mathematics & Sciences, University of New Mexico, Gallup, NM 87301, USA; smarand@unm.edu.

Abstract

The Neutrosophic Set is a mathematical framework designed to manage uncertainty, characterized by three membership functions: truth (T), indeterminacy (I), and falsity (F). In recent years, extensions such as the Hyperneutrosophic Set and SuperHyperneutrosophic Set have been introduced to address more complex scenarios. This paper proposes new concepts by extending Bipolar Neutrosophic Sets, Interval-Valued Neutrosophic Sets, Pythagorean Neutrosophic Sets, and Double-Valued Neutrosophic Sets using the frameworks of Hyperneutrosophic and SuperHyperneutrosophic Sets. Additionally, a brief analysis of these extended concepts is presented.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section outlines the essential concepts and definitions required for the discussions in this paper. For a more comprehensive understanding of foundational set theory, readers may consult references such as [24,40,42,45].

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To better address uncertainty and imprecision in decision-making, several set-theoretic models have been developed, including Fuzzy Sets [69–73], Neutrosophic Sets [26, 32–35, 37, 57, 58, 62], Plithogenic Sets [25,27,28,36,60,61,63], and Soft Sets [48,51].

Neutrosophic Sets extend Fuzzy Sets by introducing the concept of indeterminacy alongside truth and falsity [55–58]. This idea has been further developed into HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets to handle even more complex scenarios [25,29]. The following section provides their succinct definitions and relevant information.

Definition 1.1 (Neutrosophic Set)**.** [57,58] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 1.2 (Neutrosophic Set in Real Life: Medical Diagnosis)**.** (cf. [18,66])
Consider $X = \{\text{Patient A, Patient B, Patient C}\}$, the set of patients in a hospital. A Neutrosophic Set A is used to evaluate the presence of a disease D for each patient, where:

- $T_A(x)$ represents the degree of truth that the patient has the disease based on test results.
- $I_A(x)$ represents the degree of indeterminacy, accounting for inconclusive test results or lack of information.
- $F_A(x)$ represents the degree of falsity that the patient has the disease.

For example:

$$T_A(\text{Patient A}) = 0.8, \quad I_A(\text{Patient A}) = 0.1, \quad F_A(\text{Patient A}) = 0.1,$$

indicating that there is a high likelihood (80%) that Patient A has the disease, with minimal uncertainty (10%) and falsity (10%).

Definition 1.3 (HyperNeutrosophic Set)**.** (cf. [25,29–31,59]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 1.4 (HyperNeutrosophic Set in Real Life: Restaurant Review Analysis)**.** Consider

$$X = \{\text{Restaurant X, Restaurant Y, Restaurant Z}\}$$

, the set of restaurants. A HyperNeutrosophic Set $\tilde{A}$ maps each restaurant to subsets of $[0,1]^3$, where:

- (T, I, F) represents customer feedback in terms of truth (T) for positive reviews, indeterminacy (I) for neutral or unclear reviews, and falsity (F) for negative reviews.
- Multiple triplets can represent diverse opinions.

For example:

$$\tilde{\mu}(\text{Restaurant X}) = \{(0.9, 0.05, 0.05), (0.7, 0.2, 0.1)\},$$

indicating most customers rate it positively with slight variation in indeterminacy and falsity. Another restaurant:

$$\tilde{\mu}(\text{Restaurant Y}) = \{(0.4, 0.4, 0.2), (0.6, 0.3, 0.1)\},$$

shows mixed feedback with higher uncertainty in reviews.

Definition 1.5 (n-SuperHyperNeutrosophic Set)**.** (cf. [25, 29–31, 59]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Bipolar Hyperneutrosophic set

A Bipolar Neutrosophic Set (BNS) represents elements with positive and negative truth, indeterminacy, and falsity membership functions, handling dual perspectives [1, 6–9, 11, 13, 22, 50, 54, 65]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.1 (Bipolar Neutrosophic Set)**.** (cf. [7, 11, 54]) Let X be a non-empty set. A *Bipolar Neutrosophic Set (BNS) A* in X is defined as:

$$A = \left\{ \langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) \rangle : x \in X \right\},$$

where:

- $T^+, I^+, F^+ : X \to [0,1]$ are the positive truth-membership, indeterminacy-membership, and falsity-membership functions, respectively.
- $T^-, I^-, F^- : X \to [-1,0]$ are the negative truth-membership, indeterminacy-membership, and falsity-membership functions, respectively.

Here:

- $T^+(x), I^+(x), F^+(x)$ represent the degrees of truth, indeterminacy, and falsity for an element x in relation to a positive property.
- $T^-(x), I^-(x), F^-(x)$ represent the degrees of truth, indeterminacy, and falsity for an element x in relation to an implicit counter-property.

Definition 2.2 (Bipolar Hyperneutrosophic Set (BHNS))**.** Let X be a non-empty set. A *Bipolar Hyperneutrosophic Set $\widetilde{B}$* on X is a mapping

$$\widetilde{B} : X \to \mathcal{P}\Big([0,1]^3 \times [-1,0]^3 \Big),$$

such that for every $x \in X$, $\widetilde{B}(x)$ is a non-empty subset of $[0,1]^3 \times [-1,0]^3$ whose generic element can be written as $\big((T^+, I^+, F^+), (T^-, I^-, F^-)\big)$, subject to:

$$\begin{aligned} 0 \;&\leq\; T^+ + I^+ + F^+ \;\leq\; 3, \\ -3 \;&\leq\; T^- + I^- + F^- \;\leq\; 0. \end{aligned}$$

Here:

- $(T^+, I^+, F^+) \in [0,1]^3$ quantifies the positive truth, indeterminacy, and falsity for x,
- $(T^-, I^-, F^-) \in [-1,0]^3$ quantifies the negative truth, indeterminacy, and falsity for x,
- each $x \in X$ may have multiple such pairs in $\widetilde{B}(x)$, reflecting a *set-valued* or *hyper* perspective of bipolar neutrosophic membership.

Theorem 2.3. *Every Bipolar Neutrosophic Set is a special case of a Bipolar Hyperneutrosophic Set.*

Proof. A Bipolar Neutrosophic Set A on X associates each $x \in X$ with exactly one 6-tuple

$$\big(T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\big)$$

. We can embed this in Definition 2.2 by letting

$$\widetilde{B}(x) \;=\; \Big\{ \big(\big(T^+(x), I^+(x), F^+(x)\big), \big(T^-(x), I^-(x), F^-(x)\big) \big) \Big\} \subseteq [0,1]^3 \times [-1,0]^3.$$

Hence, each x maps to a *singleton set* containing the same 6-tuple from the BNS context. The constraints on $T^+ + I^+ + F^+$ and $T^- + I^- + F^-$ remain unchanged. Consequently, every BNS is realized as a special (single-valued) case of a BHNS. □

Theorem 2.4. *Every Hyperneutrosophic Set can be regarded as a special case of a Bipolar Hyperneutrosophic Set by nullifying its "negative" side.*

Proof. A Hyperneutrosophic Set $\tilde{A}$ has $\tilde{A}(x) \subseteq [0,1]^3$ with the condition $0 \le T + I + F \le 3$. In a BHNS, each $\widetilde{B}(x)$ is a subset of $[0,1]^3 \times [-1,0]^3$. If we force each (T^-, I^-, F^-) to be identically $(0,0,0)$, we essentially collapse the negative dimension. Define

$$\widetilde{B}(x) = \Big\{ \big((T, I, F), (0,0,0)\big) : (T, I, F) \in \tilde{A}(x) \Big\}.$$

Thus, $\widetilde{B}(x)$ only varies in the first (positive) triplet, effectively matching the Hyperneutrosophic membership. All conditions remain consistent, and no negativity is introduced. This recovers the exact structure of an HNS as a special BHNS case. □

Definition 2.5 (Bipolar n-SuperHyperneutrosophic Set (B-n-SHNS))**.** Let X be a non-empty set, and consider the nested power sets $\mathcal{P}_n(X)$ defined by

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \ge 2.$$

Similarly, let

$$\mathcal{P}_n\Big([0,1]^3 \times [-1,0]^3\Big)$$

denote the n-nested family of non-empty subsets of the product space $[0,1]^3 \times [-1,0]^3$.

A *Bipolar n-SuperHyperneutrosophic Set* is a mapping

$$\widetilde{B}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\Big([0,1]^3 \times [-1,0]^3\Big),$$

such that for any $A \in \mathcal{P}_n(X)$, $\widetilde{B}_n(A)$ is a (non-empty) subset of $[0,1]^3 \times [-1,0]^3$-valued "degrees of bipolar neutrosophic membership" satisfying the constraints:

$$\begin{array}{rcccl} 0 & \le & T^+ + I^+ + F^+ & \le & 3, \\ -3 & \le & T^- + I^- + F^- & \le & 0, \end{array} \quad \text{for each } \big((T^+, I^+, F^+), (T^-, I^-, F^-)\big) \in \widetilde{B}_n(A).$$

In other words, each A at the n-th nesting level is assigned a set of 6-tuples combining positive and negative membership, and each 6-tuple is bounded by the usual neutrosophic constraints of total membership in $[0,3]$ for positivity and $[-3,0]$ for negativity.

Theorem 2.6. *Every Bipolar Hyperneutrosophic Set is a particular case of a Bipolar n-SuperHyperneutrosophic Set (B-n-SHNS).*

Proof. A Bipolar Hyperneutrosophic Set $\widetilde{B}$, as defined in Definition 2.2, deals with elements $x \in X$ (so basically $n = 1$). In a B-n-SHNS from Definition 2.5, let $n = 1$, so $\mathcal{P}_1(X) = \mathcal{P}(X)$, but we only ever evaluate $\widetilde{B}_n(\{x\})$ for singletons $\{x\} \subseteq X$. Define

$$\widetilde{B}_1\big(\{x\}\big) := \widetilde{B}(x),$$

and $\widetilde{B}_1(A) := \emptyset$ (or some consistent assignment) for any $A \subseteq X$ with $|A| \ne 1$. Under this construction, we preserve all bipolarly hyperneutrosophic membership values from $\widetilde{B}$. Hence, the B-n-SHNS with $n = 1$ exactly replicates the BHNS membership in the special case where $A = \{x\}$. Therefore, any BHNS is embedded in a B-1-SHNS as a restricted scenario. □

Theorem 2.7. *Every n-SuperHyperneutrosophic Set is a special case of a Bipolar n-SuperHyperneutrosophic Set, obtained by nullifying negative membership.*

Proof. An n-SuperHyperneutrosophic Set [62] is a mapping

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

satisfying $0 \leq T+I+F \leq 3$ for each $(T, I, F) \in \tilde{A}_n(A)$. In the Bipolar n-SuperHyperneutrosophic Set context, each $\widetilde{B}_n(A)$ is a subset of $([0,1]^3 \times [-1,0]^3)$. We can force the negative part to be $(0,0,0)$, similarly to Theorem 2.4. Concretely, define

$$\widetilde{B}_n(A) = \Big\{\big((T, I, F), (0,0,0)\big) : (T, I, F) \in \tilde{A}_n(A)\Big\}.$$

All constraints remain satisfied: $0 \leq T + I + F \leq 3$ is preserved, and $T^- + I^- + F^- = 0$ lies in $[-3, 0]$. Thus, each n-SuperHyperneutrosophic membership is recovered from a Bipolar n-SuperHyperneutrosophic membership by ignoring negativity. Consequently, we obtain an n-SuperHyperneutrosophic Set as a special case of B-n-SHNS by nullifying the negative portion. □

2.2 Pythagorean Neutrosophic Set

A Pythagorean Neutrosophic Set defines truth, indeterminacy, and falsity degrees for elements, satisfying a squared-sum constraint [2–5, 14, 19, 20, 41, 52, 53]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.8 (Pythagorean Neutrosophic Set)**.** (cf. [2, 3, 19]) Let X be a non-empty set (universe). A *Pythagorean Neutrosophic Set (PNS) A* on X is defined as:

$$A = \{\langle x, u_A(x), \zeta_A(x), v_A(x)\rangle : x \in X\},$$

where:

- $u_A(x), \zeta_A(x), v_A(x) \in [0, 1]$ for all $x \in X$,
- $u_A(x), v_A(x)$ are dependent components (membership and non-membership degrees),
- $\zeta_A(x)$ is an independent component (indeterminacy degree), and
- the following condition holds:

$$0 \leq (u_A(x))^2 + (\zeta_A(x))^2 + (v_A(x))^2 \leq 2, \quad \forall x \in X.$$

Definition 2.9 (Pythagorean Hyperneutrosophic Set (PHNS))**.** Let X be a non-empty set. A *Pythagorean Hyperneutrosophic Set (PHNS) $\widetilde{A}$* on X is a mapping

$$\widetilde{A} : X \to \mathcal{P}\big([0,1]^3\big),$$

such that for each $x \in X$, the image $\widetilde{A}(x)$ is a non-empty subset of $[0, 1]^3$ whose generic element is a triplet (T, I, F) satisfying both

$$0 \leq T + I + F \leq 3 \quad \text{(the usual neutrosophic/hyperneutrosophic constraint),}$$

and the *Pythagorean* constraint

$$(T)^2 + (I)^2 + (F)^2 \leq 2.$$

Hence, each $x \in X$ is associated with multiple Pythagorean neutrosophic membership triplets in a *set-valued* manner.

Theorem 2.10 (PHNS Generalizes PNS)**.** *Every Pythagorean Neutrosophic Set is a special case of a Pythagorean Hyperneutrosophic Set.*

Proof. A *Pythagorean Neutrosophic Set (PNS)* on X is given by

$$A = \Big\{\langle x, u_A(x), \zeta_A(x), v_A(x)\rangle : x \in X\Big\},$$

with each $(u_A(x))^2 + (\zeta_A(x))^2 + (v_A(x))^2 \leq 2$ and all components in $[0, 1]$. To embed this into Definition 2.9, define a mapping $\widetilde{A}$ by:

$$\widetilde{A}(x) = \Big\{\big(u_A(x),\ \zeta_A(x),\ v_A(x)\big)\Big\} \subseteq [0, 1]^3.$$

Hence, for each $x \in X$, $\widetilde{A}(x)$ is a *singleton set* containing exactly one triplet. Clearly, $u_A(x)+\zeta_A(x)+\nu_A(x) \leq 3$ and $u_A(x)^2+\zeta_A(x)^2+\nu_A(x)^2 \leq 2$ are satisfied by assumption. Therefore, each single triplet meets the required conditions:

$$0 \leq u_A(x) + \zeta_A(x) + \nu_A(x) \leq 3, \quad (u_A(x))^2 + (\zeta_A(x))^2 + (\nu_A(x))^2 \leq 2.$$

Thus, $(X, \widetilde{A})$ is a Pythagorean Hyperneutrosophic Set that *coincides* with the given PNS in a single-valued manner. This shows every PNS is a special (singleton-valued) case of a PHNS. □

Theorem 2.11 (PHNS Generalizes HNS)**.** *Every Hyperneutrosophic Set is a special case of a Pythagorean Hyperneutrosophic Set by dropping the Pythagorean constraint.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{\mu}$ satisfies

$$\tilde{\mu}(x) \subseteq \{(T, I, F) \in [0,1]^3 : 0 \leq T + I + F \leq 3\}.$$

In Definition 2.9, we have the additional constraint $(T)^2 + (I)^2 + (F)^2 \leq 2$. If we *omit* or do not enforce $(T)^2 + (I)^2 + (F)^2 \leq 2$, we recover a standard HNS structure: let

$$\widetilde{A}(x) = \tilde{\mu}(x) \quad \text{for all } x \in X,$$

and ignore the Pythagorean condition. This matches exactly the hyperneutrosophic membership sets in $[0,1]^3$ with $T + I + F \leq 3$, thus reproducing an HNS. Hence the PHNS concept, with the Pythagorean constraint relaxed, coincides with a standard HNS. This shows HNS is strictly contained within PHNS if the Pythagorean constraint is optional. □

Definition 2.12 (Pythagorean n-SuperHyperneutrosophic Set (P-n-SHNS))**.** Let X be a non-empty set, and recall the recursively defined families:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \geq 2.$$

Similarly define $\mathcal{P}_n([0,1]^3)$ for the nested power sets of $[0,1]^3$.

A *Pythagorean n-SuperHyperneutrosophic Set (P-n-SHNS)* is a mapping

$$\widetilde{B}_n : \mathcal{P}_n(X) \rightarrow \mathcal{P}_n\big([0,1]^3\big),$$

such that for any $A \in \mathcal{P}_n(X)$, each triplet $(T, I, F) \in \widetilde{B}_n(A)$ satisfies:

$$\begin{aligned} 0 \ &\leq \ T + I + F \ \leq \ 3, \\ (T)^2 + (I)^2 + (F)^2 \ &\leq \ 2. \end{aligned}$$

In other words, each n-th level subset A is assigned a set of Pythagorean membership triplets in $[0,1]^3$, each fulfilling the usual neutrosophic/hyperneutrosophic boundary plus the Pythagorean condition.

Theorem 2.13 (P-n-SHNS Generalizes PHNS)**.** *Any Pythagorean Hyperneutrosophic Set is a particular case of a Pythagorean n-SuperHyperneutrosophic Set.*

Proof. A *Pythagorean Hyperneutrosophic Set (PHNS)* $\widetilde{A}$ assigns each $x \in X$ a subset $\widetilde{A}(x) \subseteq [0,1]^3$ of triplets fulfilling $T + I + F \leq 3$ and $T^2 + I^2 + F^2 \leq 2$. In a P-n-SHNS from Definition 2.12, choose $n = 1$, so $\mathcal{P}_1(X) = \mathcal{P}(X)$. We can define

$$\widetilde{B}_1(\{x\}) := \widetilde{A}(x)$$

and assign $\widetilde{B}_1(A) := \varnothing$ (or some consistent choice) for any $A \subseteq X$ with $|A| \neq 1$. In that case, for singletons $A = \{x\}$, we replicate precisely the membership sets from the PHNS. The constraints $T^2 + I^2 + F^2 \leq 2$ and $T + I + F \leq 3$ remain the same. Hence, $\widetilde{B}_1$ is exactly the given PHNS in restricted form. This shows any PHNS is realized as a special ($n = 1$) instance of a P-n-SHNS. □

Theorem 2.14 (P-n-SHNS Generalizes n-SHNS)**.** *Every n-SuperHyperneutrosophic Set is a special case of a Pythagorean n-SuperHyperneutrosophic Set if we discard the Pythagorean constraint.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ satisfies $T + I + F \leq 3$ for all triplets $(T, I, F) \in \tilde{A}_n(A)$, where $A \in \mathcal{P}_n(X)$. Compare this with Definition 2.12, which adds $(T)^2 + (I)^2 + (F)^2 \leq 2$. If we simply do *not* enforce the Pythagorean condition, we reproduce the original n-SHNS constraints. Formally, for a given $\tilde{A}_n$, define

$$\widetilde{B}_n(A) := \tilde{A}_n(A) \quad \text{for all } A \in \mathcal{P}_n(X),$$

and do *not* impose $(T)^2 + (I)^2 + (F)^2 \leq 2$. Then $\widetilde{B}_n$ matches exactly the membership sets from $\tilde{A}_n$. Therefore, ignoring Pythagorean constraints yields an ordinary n-SHNS. Consequently, any n-SHNS is a special case of a P-n-SHNS in which we disregard the extra Pythagorean condition. □

2.3 Double-Valued Neutrosophic Set

A Double-Valued Neutrosophic Set represents truth, indeterminacy (toward truth/falsity), and falsity degrees for elements, summing up to ≤ 4 [23,38,39,43,44,46,47,67,76,77]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.15 (Double-Valued Neutrosophic Set)**.** [43] Let X be a non-empty set (universe). A *Double-Valued Neutrosophic Set (DVNS) A* on X is defined as:

$$A = \{\langle x, T_A(x), I_T(x), I_F(x), F_A(x)\rangle : x \in X\},$$

where:

- $T_A(x), I_T(x), I_F(x), F_A(x) \in [0, 1]$ for all $x \in X$,
- $T_A(x)$: truth membership degree,
- $I_T(x)$: indeterminacy leaning towards truth,
- $I_F(x)$: indeterminacy leaning towards falsity,
- $F_A(x)$: falsity membership degree,
- the following condition holds:

$$0 \leq T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4, \quad \forall x \in X.$$

Definition 2.16 (Double-Valued Hyperneutrosophic Set (DVHNS))**.** Let X be a non-empty set. A *Double-Valued Hyperneutrosophic Set $\widetilde{D}$* on X is a mapping

$$\widetilde{D} : X \to \mathcal{P}([0, 1]^4),$$

where $\mathcal{P}([0, 1]^4)$ denotes the family of all non-empty subsets of $[0, 1]^4$. For each $x \in X$, the set $\widetilde{D}(x) \subseteq [0, 1]^4$ consists of quadruples $\big(T, I_T, I_F, F\big)$ that satisfy

$$0 \leq T + I_T + I_F + F \leq 4.$$

Here:

- T = truth-membership degree,
- I_T = indeterminacy leaning towards truth,
- I_F = indeterminacy leaning towards falsity,
- F = falsity-membership degree.

Each $x \in X$ may be associated with *multiple* such quadruples, forming a *set-valued* membership structure.

Theorem 2.17. *Every Double-Valued Neutrosophic Set is a special case of a Double-Valued Hyperneutrosophic Set.*

Proof. A *Double-Valued Neutrosophic Set (DVNS) A* on X assigns each $x \in X$ a unique 4-tuple

$$\big(T_A(x), I_T(x), I_F(x), F_A(x)\big)$$

with $T_A + I_T + I_F + F_A \leq 4$. We embed this into Definition 2.16 by letting

$$\widetilde{D}(x) = \Big\{\big(T_A(x),\ I_T(x),\ I_F(x),\ F_A(x)\big)\Big\} \ \subseteq\ [0,1]^4.$$

Hence, each x is mapped to a *singleton set* containing precisely the same quadruple. The condition $T_A + I_T + I_F + F_A \leq 4$ remains unchanged. Therefore, each single-valued DVNS is captured as a special (singleton) DVHNS. □

Theorem 2.18. *Every Hyperneutrosophic Set is a special case of a Double-Valued Hyperneutrosophic Set when the extra (fourth) component is nullified.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{\mu}$ satisfies

$$\tilde{\mu}(x) \subseteq \Big\{(T, I, F) \in [0,1]^3 : 0 \leq T + I + F \leq 3\Big\}.$$

In Definition 2.16, each membership is a subset of $[0,1]^4$ with $T + I_T + I_F + F \leq 4$. We reduce to HNS by forcing $I_F = 0$. Concretely, define

$$\widetilde{D}(x) = \Big\{(T, I, 0, F)\ \mid\ (T, I, F) \in \tilde{\mu}(x)\Big\} \ \subseteq\ [0,1]^4.$$

Then the condition $T + I + 0 + F \leq 4$ is effectively $T + I + F \leq 4$. By restricting further to $T + I + F \leq 3$ (which is typically satisfied in HNS), we see that ignoring the extra dimension recovers the standard 3-component condition. Thus, HNS is included within DVHNS by identifying the extra dimension with zero. □

Definition 2.19 (Double-Valued n-SuperHyperneutrosophic Set (DV-n-SHNS))**.** Let X be a non-empty set, and let $\mathcal{P}_n(X)$ be the n-th nested power set of X, defined by:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \geq 2.$$

Similarly, define $\mathcal{P}_n([0,1]^4)$ for the nested power sets of $[0,1]^4$.

A *Double-Valued n-SuperHyperneutrosophic Set* $\widetilde{D}_n$ is a mapping:

$$\widetilde{D}_n\ :\ \mathcal{P}_n(X)\ \longrightarrow\ \mathcal{P}_n\big([0,1]^4\big),$$

such that for any $A \in \mathcal{P}_n(X)$ and for any quadruple $\big(T, I_T, I_F, F\big) \in \widetilde{D}_n(A) \subseteq [0,1]^4$, the following holds:

$$0\ \leq\ T + I_T + I_F + F\ \leq\ 4.$$

Theorem 2.20. *Every Double-Valued Hyperneutrosophic Set is a particular case of a Double-Valued n-SuperHyperneutrosophic Set.*

Proof. A *Double-Valued Hyperneutrosophic Set (DVHNS)* $\widetilde{D}$ has $\widetilde{D}(x) \subseteq [0,1]^4$ for each $x \in X$. In a DV-n-SHNS (Definition 2.19), choose $n = 1$. Then

$$\widetilde{D}_1 : \mathcal{P}_1(X) = \mathcal{P}(X)\ \to\ \mathcal{P}_1\big([0,1]^4\big) = \mathcal{P}\big([0,1]^4\big).$$

We can define:

$$\widetilde{D}_1(\{x\}) = \widetilde{D}(x), \quad \text{and possibly set } \widetilde{D}_1(A) = \varnothing \text{ for other } A \neq \{x\}.$$

Hence, restricting to singletons $\{x\} \subseteq X$ recovers exactly the DVHNS. Thus, the DVHNS is embedded in the DV-1-SHNS as a special case. □

Theorem 2.21. *Every n-SuperHyperneutrosophic Set is a special case of a Double-Valued n-SuperHyperneutrosophic Set by nullifying the extra dimension.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ maps $A \in \mathcal{P}_n(X)$ to subsets of $[0,1]^3$, each satisfying $T + I + F \leq 3$. In DV-n-SHNS, each membership is in $[0,1]^4$ with $T + I_T + I_F + F \leq 4$. To match an SHNS, we can do the following for each A:

$$\widetilde{D}_n(A) = \Big\{ \big(T, I, 0, F\big) : (T, I, F) \in \tilde{A}_n(A) \Big\}.$$

We also may require $T + I + F \leq 3$ to remain consistent, embedded in $T + I + (0) + F \leq 4$. This effectively sets $I_F = 0$, reducing the dimension to 3. Thus, ignoring the fourth component recovers the usual n-SHNS form. Therefore, every n-SHNS is embedded in DV-n-SHNS by trivializing the extra dimension. □

2.4 Interval-Valued Neutrosophic Set

An Interval-Valued Neutrosophic Set assigns interval-based truth, indeterminacy, and falsity degrees to elements, capturing uncertainty within specified ranges [10, 12, 15–17, 21, 49, 64, 68, 74, 75]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.22 (Interval-Valued Neutrosophic Set)**.** (cf. [21, 68, 74, 75]) Let X be a non-empty set (universe). An *Interval-Valued Neutrosophic Set (IVNS) A* on X is defined as:

$$A = \left\{ \langle x, [T_A^l(x), T_A^r(x)], [I_A^l(x), I_A^r(x)], [F_A^l(x), F_A^r(x)] \rangle : x \in X \right\},$$

where:

- $[T_A^l(x), T_A^r(x)]$: interval of truth membership degrees,
- $[I_A^l(x), I_A^r(x)]$: interval of indeterminacy membership degrees,
- $[F_A^l(x), F_A^r(x)]$: interval of falsity membership degrees,
- $T_A^l(x), T_A^r(x), I_A^l(x), I_A^r(x), F_A^l(x), F_A^r(x) \in [0,1]$,
- and the condition:
$$0 \leq T_A^r(x) + I_A^r(x) + F_A^r(x) \leq 3, \quad \forall x \in X.$$

Definition 2.23 (Interval-Valued Hyperneutrosophic Set (IVHNS))**.** Let X be a non-empty set. An *Interval-Valued Hyperneutrosophic Set (IVHNS) $\widetilde{H}$* on X is a mapping

$$\widetilde{H} : X \to \mathcal{P}\big([0,1]^6\big),$$

where each $\widetilde{H}(x) \subseteq [0,1]^6$ is a (non-empty) set of *interval-triplets*

$$\Big(\big(T^l, T^r\big), \big(I^l, I^r\big), \big(F^l, F^r\big) \Big),$$

subject to:

1. $0 \leq T^l \leq T^r \leq 1, \quad 0 \leq I^l \leq I^r \leq 1, \quad 0 \leq F^l \leq F^r \leq 1,$
2. The *upper bounds* satisfy:
$$T^r + I^r + F^r \quad \leq \quad 3.$$

In other words, for each $x \in X$, $\widetilde{H}(x)$ is a set of intervals describing the truth, indeterminacy, and falsity degrees in $[0,1]$ such that the sum of the *right endpoints* does not exceed 3.

Theorem 2.24 (IVHNS Generalizes IVNS)**.** *Every Interval-Valued Neutrosophic Set is a particular case of an Interval-Valued Hyperneutrosophic Set.*

Proof. An *Interval-Valued Neutrosophic Set (IVNS)* A on X is given by

$$A = \Big\{\langle x, [T_A^l(x), T_A^r(x)], [I_A^l(x), I_A^r(x)], [F_A^l(x), F_A^r(x)]\rangle \;:\; x \in X\Big\},$$

where $T_A^r(x) + I_A^r(x) + F_A^r(x) \le 3$. To embed this in Definition 2.23, define a set-valued mapping $\widetilde{H}$ by:

$$\widetilde{H}(x) = \Big\{\big((T_A^l(x),\, T_A^r(x)), (I_A^l(x),\, I_A^r(x)), (F_A^l(x),\, F_A^r(x))\big)\Big\} \subseteq [0,1]^6.$$

Hence, each x maps to a *singleton set* containing exactly one interval-triplet. The condition on the right endpoints ≤ 3 is the same. Therefore, each single-valued IVNS is captured as a special (singleton) IVHNS. □

Theorem 2.25 (IVHNS Generalizes HNS)**.** *Every Hyperneutrosophic Set is a special case of an Interval-Valued Hyperneutrosophic Set by restricting intervals to single points.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{\mu}$ assigns each $x \in X$ a subset of $[0,1]^3$, with (T, I, F) satisfying $T + I + F \le 3$. In the IVHNS of Definition 2.23, each membership is a subset of $[0,1]^6$ of interval-triplets $(T^l, T^r, I^l, I^r, F^l, F^r)$. If we *force* each pair (T^l, T^r) to collapse to (T, T), (I^l, I^r) to (I, I), and (F^l, F^r) to (F, F), then effectively

$$\widetilde{H}(x) = \Big\{\big((T,T), (I,I), (F,F)\big) : (T, I, F) \in \tilde{\mu}(x)\Big\}.$$

The sum-of-right-endpoints constraint becomes $T + I + F \le 3$. This matches the standard HNS membership. Hence, HNS emerges as a special case of IVHNS by identifying intervals with their single-point degenerate intervals. □

Definition 2.26 (Interval-Valued n-SuperHyperneutrosophic Set (IV-n-SHNS))**.** Let X be a non-empty set, and define

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \text{ for } k \ge 2.$$

Similarly, we define $\mathcal{P}_n([0,1]^6)$ for the nested power set of $[0,1]^6$. An *Interval-Valued n-SuperHyperneutrosophic Set (IV-n-SHNS)* is a mapping

$$\widetilde{H}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big([0,1]^6\big),$$

such that for any $A \in \mathcal{P}_n(X)$ and any $\big((T^l, T^r), (I^l, I^r), (F^l, F^r)\big) \in \widetilde{H}_n(A) \subseteq [0,1]^6$, the following hold:

$$0 \le T^l \le T^r \le 1, \quad 0 \le I^l \le I^r \le 1, \quad 0 \le F^l \le F^r \le 1,$$

and

$$T^r + I^r + F^r \le 3.$$

In other words, each n-th level subset $A \subseteq X$ is assigned a *set* of interval-triplets $\big([T^l, T^r], [I^l, I^r], [F^l, F^r]\big)$ satisfying the usual neutrosophic upper-bound constraint on $(T^r + I^r + F^r)$.

Theorem 2.27. *Every Interval-Valued Hyperneutrosophic Set is a special case of an Interval-Valued n-SuperHyperneutrosophic Set.*

Proof. An *Interval-Valued Hyperneutrosophic Set (IVHNS)* $\widetilde{H}$ is a mapping $\widetilde{H} : X \to \mathcal{P}([0,1]^6)$. In IV-$n$-SHNS (Definition 2.26), choose $n = 1$. Then:

$$\widetilde{H}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \to \mathcal{P}_1\big([0,1]^6\big) = \mathcal{P}\big([0,1]^6\big).$$

Define

$$\widetilde{H}_1(\{x\}) = \widetilde{H}(x), \quad \text{and possibly set } \widetilde{H}_1(A) = \varnothing \text{ for } A \ne \{x\}.$$

Hence, restricting to singletons recovers the IVHNS exactly. Thus, an IVHNS is embedded in IV-1-SHNS as a special case. □

Theorem 2.28. *Every n-SuperHyperneutrosophic Set is a special case of an Interval-Valued n-SuperHyperneutrosophic Set by making each interval degenerate to a point.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ maps $A \in \mathcal{P}_n(X)$ to subsets of $[0,1]^3$, each triplet (T, I, F) with $T + I + F \leq 3$. The IV-n-SHNS in Definition 2.26 uses subsets of $[0,1]^6$ representing intervals $\left(T^l, T^r, I^l, I^r, F^l, F^r\right)$. We can force each interval to collapse to a single point:

$$T^l = T^r = T, \quad I^l = I^r = I, \quad F^l = F^r = F,$$

where $(T, I, F) \in [0,1]^3$. Define

$$\widetilde{H}_n(A) = \Big\{\big((T,T),(I,I),(F,F)\big) : (T,I,F) \in \tilde{A}_n(A)\Big\}.$$

Hence, $T^r + I^r + F^r = T + I + F \leq 3$ becomes the standard condition. Therefore, ignoring the interval nature yields an n-SHNS. Consequently, any n-SHNS is subsumed under IV-n-SHNS by setting intervals to degenerate points. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Mohamed Abdel-Basset, Mai Mohamed, Mohamed Elhoseny, Le Hoang Son, Francisco Chiclana, and Abdel Nasser H. Zaied. Cosine similarity measures of bipolar neutrosophic set for diagnosis of bipolar disorder diseases. *Artificial intelligence in medicine*, 101:101735, 2019.

[2] Noraini Ahmad, Zahari Rodzi, Faisal Al-Sharqi, Ashraf Al-Quran, Abdalwali Lutfi, Zanariah Mohd Yusof, Nor Aini Hassanuddin, et al. Innovative theoretical approach: Bipolar pythagorean neutrosophic sets (bpnss) in decision-making. *Full Length Article*, 23(1):249–49, 2023.

[3] D. Ajay and P. Chellamani. Pythagorean neutrosophic soft sets and their application to decision-making scenario. *Intelligent and Fuzzy Techniques for Emerging Conditions and Digital Transformation*, 2021.

[4] D. Ajay, S. John Borg, and P. Chellamani. Domination in pythagorean neutrosophic graphs with an application in fuzzy intelligent decision making. In *International Conference on Intelligent and Fuzzy Systems*, pages 667–675, Cham, July 2022. Springer International Publishing.

[5] D Ajay, S Karthiga, and P Chellamani. A study on labelling of pythagorean neutrosophic fuzzy graphs. *Journal of Computational Mathematica*, 5(1):105–116, 2021.

[6] Muhammad Akram. Certain bipolar neutrosophic competition graphs. *Journal of the Indonesian Mathematical Society*, 24:1–25, 2018.

[7] Muhammad Akram and Anam Luqman. Bipolar neutrosophic hypergraphs with applications. *Journal of Intelligent & Fuzzy Systems*, 33(3):1699–1713, 2017.

[8] Muhammad Akram and Anam Luqman. Bipolar neutrosophic hypergraphs with applications. *J. Intell. Fuzzy Syst.*, 33:1699–1713, 2017.

[9] Muhammad Akram and Anam Luqman. A new decision-making method based on bipolar neutrosophic directed hypergraphs. *Journal of Applied Mathematics and Computing*, 57:547 – 575, 2017.

[10] Muhammad Akram and Maryam Nasir. *Interval-valued neutrosophic competition graphs*. Infinite Study, 2017.

[11] Muhammad Akram, Shumaiza, and Florentin Smarandache. Decision-making with bipolar neutrosophic topsis and bipolar neutrosophic electre-i. *Axioms*, 7:33, 2018.

[12] Faisal Al-Sharqi, Abd Ghafur Ahmad, and Ashraf Al-Quran. Interval-valued neutrosophic soft expert set from real space to complex space. *Computer Modeling in Engineering & Sciences*, 2022.

[13] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *Journal of Intelligent & Fuzzy Systems*, 33(6):4077–4087, 2017.

[14] Mohammad Shafiq bin Mohammad Kamari, Zahari Bin Md Rodzi, RH Al-Obaidi, Faisal Al-Sharq, Ashraf Al-Quran, et al. Deciphering the geometric bonferroni mean operator in pythagorean neutrosophic sets framework. *Neutrosophic Sets and Systems*, 75:139–161, 2025.

[15] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and Prem Kumar Singh. Properties of interval-valued neutrosophic graphs. *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, pages 173–202, 2019.

[16] Said Broumi and Florentin Smarandache. Interval-valued neutrosophic soft rough sets. 2015.

[17] Said Broumi, Mohamed Talea, Assia Bakali, Florentin Smarandache, Quek Shio Gai, and Ganeshsree Selvachandran. Introduction of some new results on interval-valued neutrosophic graphs. *Journal of Information and Optimization Sciences*, 40(7):1475–1498, 2019.

[18] Quang-Thinh Bui, My-Phuong Ngo, Vaclav Snasel, Witold Pedrycz, and Bay Vo. The sequence of neutrosophic soft sets and a decision-making problem in medical diagnosis. *International Journal of Fuzzy Systems*, 24:2036 – 2053, 2022.

[19] P Chellamani and D Ajay. Pythagorean neutrosophic dombi fuzzy graphs with an application to mcdm. *Neutrosophic Sets and Systems*, 47:411–431, 2021.

[20] P Chellamani, D Ajay, Mohammed M Al-Shamiri, and Rashad Ismail. *Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making*. Infinite Study, 2023.

[21] Irfan Deli. Interval-valued neutrosophic soft sets and its decision making. *International Journal of Machine Learning and Cybernetics*, 8:665–676, 2017.

[22] Irfan Deli, S Yusuf, Florentin Smarandache, and Mumtaz Ali. Interval valued bipolar neutrosophic sets and their application in pattern recognition. In *IEEE world congress on computational intelligence*, 2016.

[23] Kanghua Du and Yuming Du. Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:2813–2822, 2023.

[24] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[25] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[26] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[27] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[28] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[29] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[30] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[31] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[32] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[33] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[34] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[35] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[36] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[37] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[38] Yourong Guo. Enhancing dvnn-wcsm technique for double-valued neutrosophic multiple-attribute decision-making in digital economy: A case study on enhancing the quality of development of henan's cultural and tourism industry. *Neutrosophic Sets and Systems*, 75:391–407, 2025.

[39] Heng He. A novel approach to assessing art education teaching quality in vocational colleges based on double-valued neutrosophic numbers and multi-attribute decision-making with tree soft sets. *Neutrosophic Sets and Systems*, 78:206–218, 2025.

[40] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[41] Jamiatun Nadwa Ismail, Zahari Rodzi, Faisal Al-Sharqi, Ashraf Al-Quran, Hazwani Hashim, and Nor Hashimah Sulaiman. Algebraic operations on pythagorean neutrosophic sets (pns): Extending applicability and decision-making capabilities. *International Journal of Neutrosophic Science (IJNS)*, 21(4), 2023.

[42] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[43] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[44] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[45] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[46] Peng Liu and Xiaonan Geng. Evaluation model of green supplier selection for coal enterprises with similarity measures of double-valued neutrosophic sets based on cosine function. *Journal of Intelligent & Fuzzy Systems*, 44(6):9257–9265, 2023.

[47] M. M., M. Suneetha, Maria Mikhailova, Sripada Nsvsc Ramesh, and Kollati Vijaya Kumar. Leveraging double-valued neutrosophic set for real-time chronic kidney disease detection and classification. *International Journal of Neutrosophic Science*.

[48] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[49] Siti Nurul Fitriah Mohamad, Roslan Hasni, Florentin Smarandache, and Binyamin Yusoff. Novel concept of energy in bipolar single-valued neutrosophic graphs with applications. *Axioms*, 10(3):172, 2021.

[50] Mai Mohamed and Asmaa Elsayed. A novel multi-criteria decision making approach based on bipolar neutrosophic set for evaluating financial markets in egypt. *Multicriteria Algorithms with Applications*, 2024.

[51] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[52] Anjan Mukherjee and Rakhal Das. Application of pythagorean neutrosophic vague soft on decision making problem. *New Trends in Neutrosophic Theories and Applications*, page 9.

[53] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[54] Prem Kumar Singh. Three-way bipolar neutrosophic concept lattice. *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, 2018.

[55] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[56] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[57] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[58] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[59] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[60] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[61] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[62] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[63] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[64] Keneni Abera Tola, VN Srinivasa Rao Repalle, and Mamo Abebe Ashebo. Theory and application of interval-valued neutrosophic line graphs. *Journal of Mathematics*, 2024(1):5692756, 2024.

[65] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[66] Vakkas Ulucay, Adil Kılıç, Memet Sahin, and Harun Deniz. *A new hybrid distance-based similarity measure for refined neutrosophic sets and its application in medical diagnosis*. Infinite Study, 2019.

[67] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[68] Jun Ye and Shigui Du. Some distances, similarity and entropy measures for interval-valued neutrosophic sets and their relationship. *International Journal of Machine Learning and Cybernetics*, 10:347 – 355, 2017.

[69] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[70] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[71] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[72] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[73] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[74] Hongyu Zhang, Jian qiang Wang, and Xiao hong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615 – 627, 2015.

[75] Hongyu Zhang, Jianqiang Wang, and Xiaohong Chen. An outranking approach for multi-criteria decision-making problems with interval-valued neutrosophic sets. *Neural Computing and Applications*, 27:615–627, 2016.

[76] Zhiheng Zhang. Enhanced decision-making technique for innovation capability evaluation in the core industries of digital economy under double-valued neutrosophic sets. *Neutrosophic Sets and Systems*, 77:492–509, 2025.

[77] Qiuyan Zhao and Wentao Li. Incorporating intelligence in multiple-attribute decision-making using algorithmic framework and double-valued neutrosophic sets: Varied applications to employment quality evaluation for university graduates. *Neutrosophic Sets and Systems*, 76:59–78, 2025.

Chapter 4
Some Types of HyperNeutrosophic Set (2): Complex, Single-Valued Triangular, Fermatean, and Linguistic Sets

Takaaki Fujita [1 *] and Florentin Smarandache[2],
[1*] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp
[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

This paper is a continuation of the work presented in [35]. The Neutrosophic Set provides a mathematical framework for managing uncertainty, characterized by three membership functions: truth, indeterminacy, and falsity. Recent advancements have introduced extensions such as the Hyperneutrosophic Set and SuperHyperneutrosophic Set to address more complex and multidimensional challenges.

In this study, we extend the Complex Neutrosophic Set, Single-Valued Triangular Neutrosophic Set, Fermatean Neutrosophic Set, and Linguistic Neutrosophic Set within the frameworks of Hyperneutrosophic Sets and SuperHyperneutrosophic Sets. Furthermore, we investigate their mathematical structures and analyze their connections with other set-theoretic concepts.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section outlines the essential concepts and definitions required for the discussions in this paper. Basic set operations are utilized throughout this study. For a more comprehensive understanding of foundational set theory, readers are encouraged to consult references such as [24, 46, 48, 51]. Additionally, for fundamental operations and applications of Neutrosophic Sets, the relevant literature should be referred to as cited.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To effectively address uncertainty and imprecision in decision-making, various set-theoretic frameworks have been developed. These include Fuzzy Sets [77–81], Intuitionistic Fuzzy Sets [7–12], Soft sets [45, 54, 55], and Neutrosophic Sets [26, 27, 34, 36–41, 43, 44, 64, 65, 69]. Additionally, advanced extensions such as Plithogenic Sets [25, 28–30, 42, 67, 68, 70] and Rough Sets [56–60] have been proposed. These models expand upon traditional set theories to better capture complex, multidimensional, and uncertain decision-making scenarios. Neutrosophic Sets extend Fuzzy Sets by introducing the concept of indeterminacy alongside truth and falsity [62–65]. This idea has been further developed into HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets to handle even more complex scenarios [25, 31]. The following section provides their succinct definitions and relevant information.

Definition 1.1 (Neutrosophic Set)**.** [64, 65] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 1.2 (HyperNeutrosophic Set)**.** (cf. [25, 31–33, 66]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \le T + I + F \le 3.$$

Definition 1.3 (n-SuperHyperNeutrosophic Set)**.** (cf. [25, 31–33, 66]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

$$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Complex Neutrosophic Set

A Complex Neutrosophic Set represents truth, indeterminacy, and falsity memberships as points on the complex unit circle [4–6, 13, 14, 53, 72]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.1 (Complex Neutrosophic Set)**.** [4, 5] Let U be a universe of discourse, and let $u \in U$ be an element. A *Complex Neutrosophic Set A* on U is characterized by three membership functions:

- *Truth-membership function* $T_A(u)$,
- *Indeterminacy-membership function* $I_A(u)$,
- *Falsity-membership function* $F_A(u)$,

where each function maps to the complex unit circle $\mathbb{C}_1$, i.e.,

$$T_A(u), I_A(u), F_A(u) \in \mathbb{C}_1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

The set A is represented as:

$$A = \{\langle u, T_A(u), I_A(u), F_A(u)\rangle \mid u \in U\}.$$

Definition 2.2 (Complex Hyperneutrosophic Set (CHNS))**.** Let X be a non-empty set, and let $\mathcal{P}(\mathcal{S})$ denote the family of all non-empty subsets of some region $\mathcal{S} \subseteq \mathbb{C}^3$. A *Complex Hyperneutrosophic Set (CHNS) $\widetilde{C}$* on X is a mapping

$$\widetilde{C} : X \to \mathcal{P}(\mathcal{S}),$$

where each $\mathcal{S}$ is a subset of $\mathbb{C}^3$ that encodes the neutrosophic constraint in the complex domain. Concretely, for each $x \in X$,

$$\widetilde{C}(x) \subseteq \{(T, I, F) \in \mathcal{S} \subseteq \mathbb{C}^3 : |T| + |I| + |F| \leq 3\},$$

where (T, I, F) are the *complex truth, indeterminacy, and falsity values* for x. One might specifically choose $\mathcal{S} = \{(T, I, F) \in \mathbb{C}^3 : |T|, |I|, |F| \leq 1\}$ to mimic the real case.

Hence, each element $x \in X$ may have a *set* of possible complex neutrosophic membership triplets (T, I, F). Each such triplet satisfies:

$$|T| + |I| + |F| \leq 3.$$

Theorem 2.3. *Every Complex Neutrosophic Set is a special case of a Complex Hyperneutrosophic Set.*

Proof. A *Complex Neutrosophic Set (CNS) A* on X associates each $x \in X$ with exactly one triplet $(T_A(x), I_A(x), F_A(x)) \in \mathcal{S} \subseteq \mathbb{C}^3$, typically satisfying $|T_A(x)| + |I_A(x)| + |F_A(x)| \leq 3$. We embed this in Definition 2.2 by letting

$$\widetilde{C}(x) = \Big\{\big(T_A(x),\ I_A(x),\ F_A(x)\big)\Big\},$$

a singleton in $\mathcal{P}(\mathcal{S})$. The same magnitude constraint holds, so every CNS is realized as a single-valued (singleton) Complex Hyperneutrosophic Set. □

Theorem 2.4. *Every (real) Hyperneutrosophic Set is a special case of a Complex Hyperneutrosophic Set, by restricting complex values to real numbers.*

Proof. A standard *Hyperneutrosophic Set (HNS)* $\tilde{A}$ satisfies

$$\tilde{A}(x) \ \subseteq \ \Big\{(T, I, F) \in [0,1]^3 : T + I + F \leq 3\Big\}.$$

In Definition 2.2, each membership is $\widetilde{C}(x) \subseteq \mathcal{S} \subseteq \mathbb{C}^3$. To match the real setting, force

$$T, I, F \ \in \ [0,1] \subset \mathbb{C}, \quad \text{(imaginary part = 0)},$$

and impose $|T| + |I| + |F| = T + I + F \leq 3$. Then

$$\widetilde{C}(x) = \Big\{(T, I, F) \in [0,1]^3 : T + I + F \leq 3\Big\}.$$

Hence we recover the real HNS as a restriction of CHNS to purely real triplets. Therefore, every real HNS is embedded in a CHNS by ignoring any imaginary component. □

Definition 2.5 (Complex n-SuperHyperneutrosophic Set (C-n-SHNS))**.** Let X be a non-empty set, and define

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad \text{for } k \geq 2.$$

Similarly, let $\mathcal{P}_n(\mathcal{S})$ denote the n-th nested power set of a region $\mathcal{S} \subseteq \mathbb{C}^3$. A *Complex n-SuperHyperneutrosophic Set (C-n-SHNS)* is a mapping

$$\widetilde{C}_n : \mathcal{P}_n(X) \ \longrightarrow \ \mathcal{P}_n(\mathcal{S}),$$

such that for any $A \in \mathcal{P}_n(X)$ and any triplet $(T, I, F) \in \widetilde{C}_n(A) \subseteq \mathcal{S} \subseteq \mathbb{C}^3$, we have

$$|T| + |I| + |F| \ \leq \ 3.$$

Thus, each n-th level subset $A \subseteq X$ is assigned a *set* of complex-valued membership triplets $\big(T, I, F\big)$ satisfying the magnitude-sum condition ≤ 3, just as in the single-level CHNS, but now extended to n-nested subsets of X.

Theorem 2.6. *Every Complex Hyperneutrosophic Set is a special case of a Complex n-SuperHyperneutrosophic Set (C-n-SHNS), obtained by setting $n = 1$.*

Proof. A *Complex Hyperneutrosophic Set (CHNS)* $\widetilde{C}$ has $\widetilde{C}(x) \subseteq \mathcal{S} \subseteq \mathbb{C}^3$. In Definition 2.5, let $n = 1$. Then

$$\widetilde{C}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \ \longrightarrow \ \mathcal{P}_1(\mathcal{S}) = \mathcal{P}(\mathcal{S}).$$

We can define:

$$\widetilde{C}_1(\{x\}) := \widetilde{C}(x), \quad \text{and possibly set } \widetilde{C}_1(A) = \varnothing \text{ for } A \neq \{x\}.$$

Hence, restricting to singletons $\{x\} \subseteq X$ reproduces precisely the membership sets $\widetilde{C}(x)$. The constraint $|T| + |I| + |F| \leq 3$ persists. Therefore, any CHNS is embedded in a C-1-SHNS as a single-level scenario. □

Theorem 2.7. *Every (real) n-SuperHyperneutrosophic Set is a special case of a Complex n-SuperHyperneutrosophic Set by restricting triplets to real values.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ satisfies

$$\tilde{A}_n(A) \;\subseteq\; \{(T,I,F) \in [0,1]^3 : T+I+F \leq 3\}$$

for each $A \in \mathcal{P}_n(X)$. Compare with Definition 2.5, where

$$\widetilde{C}_n(A) \;\subseteq\; \mathcal{S} \subseteq \mathbb{C}^3 \quad \text{with} \quad |T|+|I|+|F| \leq 3.$$

Restrict each T, I, F to be real and non-negative, i.e. let

$$T, I, F \in [0,1] \subset \mathbb{C} \quad \text{(imaginary part = 0)},$$

and impose $T+I+F \leq 3$. Define

$$\widetilde{C}_n(A) = \Big\{(T,I,F) \in [0,1]^3 : T+I+F \leq 3,\ (T,I,F) \in \tilde{A}_n(A)\Big\}.$$

Hence, the real n-SuperHyperneutrosophic membership sets appear as a restriction of the complex domain. Therefore, every n-SHNS is included in a C-n-SHNS by limiting the membership triplets to real values. □

2.2 Single-valued triangular neutrosophic set

A Single-Valued Triangular Neutrosophic Set uses triangular fuzzy numbers to represent truth, indeterminacy, and falsity membership functions for elements in a set [1, 2, 20, 22, 23, 47, 49, 71]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.8 (Single-Valued Triangular Neutrosophic Set)**.** (cf. [2,23]) Let X be a universe of discourse. A *Single-Valued Triangular Neutrosophic Set (SVTNS) A* in X is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\},$$

where each of $T_A(x), I_A(x), F_A(x)$ is a *triangular fuzzy number*, expressed as:

$$T_A(x) = \langle T_l, T_m, T_r\rangle, \quad I_A(x) = \langle I_l, I_m, I_r\rangle, \quad F_A(x) = \langle F_l, F_m, F_r\rangle.$$

The real numbers (T_l, T_m, T_r), (I_l, I_m, I_r), and (F_l, F_m, F_r) satisfy $T_l \leq T_m \leq T_r$, $I_l \leq I_m \leq I_r$, $F_l \leq F_m \leq F_r$. Each triangular fuzzy membership is defined piecewise, e.g.:

$$T_A(x) = \begin{cases} \frac{x-T_l}{T_m-T_l}, & \text{if } T_l \leq x \leq T_m, \\ \frac{T_r-x}{T_r-T_m}, & \text{if } T_m \leq x \leq T_r, \\ 0, & \text{otherwise.} \end{cases}$$

Similar piecewise definitions hold for $I_A(x)$ and $F_A(x)$.

Intuitively,

- $T_A(x)$ represents the "truth-membership" of x with a triangular shape,
- $I_A(x)$ represents the "indeterminacy-membership" of x,
- $F_A(x)$ represents the "falsity-membership" of x.

This structure extends classical Single-Valued Neutrosophic Sets by allowing each membership function to follow a triangular distribution over X.

Definition 2.9 (Single-Valued Triangular Hyperneutrosophic Set (SVTHNS))**.** Let X be a non-empty set, and let

$$\mathcal{T} \;=\; \Big\{\langle t_l, t_m, t_r\rangle : t_l \leq t_m \leq t_r, \quad t_l, t_m, t_r \in \mathbb{R}\Big\}$$

be the set of all possible triangular fuzzy numbers on the real line. A *Single-Valued Triangular Hyperneutrosophic Set (SVTHNS)* $\widetilde{S}$ on X is a mapping

$$\widetilde{S} : X \longrightarrow \mathcal{P}(\mathcal{T}^3),$$

where for each $x \in X$,

$$\widetilde{S}(x) \;\subseteq\; \Big\{(T, I, F) \in \mathcal{T}^3 \;\Big|\; \text{each of } T, I, F \text{ is a triangular fuzzy number, and } \max(T_r, I_r, F_r) \le M\Big\},$$

for some real upper bound $M > 0$ (analogous to constraints like ≤ 3 in standard neutrosophic sets, though one might specify additional conditions if desired).

In simpler terms, each $x \in X$ can be assigned a *set* of triplets (T, I, F), each of which is made of triangular fuzzy numbers for truth, indeterminacy, and falsity, respectively.

Theorem 2.10. *Every Single-Valued Triangular Neutrosophic Set is a special case of a Single-Valued Triangular Hyperneutrosophic Set.*

Proof. A *Single-Valued Triangular Neutrosophic Set (SVTNS) A* on X associates each $x \in X$ with exactly one triplet of triangular fuzzy numbers $(T_A(x), I_A(x), F_A(x))$. To embed this into Definition 2.9, define

$$\widetilde{S}(x) = \Big\{(T_A(x), I_A(x), F_A(x))\Big\},$$

i.e. a singleton set. Each $(T_A(x), I_A(x), F_A(x)) \in \mathcal{T}^3$ is already guaranteed by the triangular membership definitions for truth, indeterminacy, and falsity. Thus, every SVTNS is reproduced as a Single-Valued Triangular Hyperneutrosophic Set with singletons. □

Theorem 2.11. *Every (real) Hyperneutrosophic Set is a special case of a Single-Valued Triangular Hyperneutrosophic Set, by restricting the triangular fuzzy numbers to degenerate points.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{A}$ maps each $x \in X$ to a (possibly multi-valued) subset of $[0, 1]^3$, e.g. triplets $(T, I, F) \in [0, 1]^3$. In Definition 2.9, each membership is a subset of $\mathcal{T}^3$. To mimic the real HNS scenario, let each triangular fuzzy number degenerate to a single real point, i.e.

$$\langle t_l, t_m, t_r \rangle = \langle r, r, r \rangle, \quad r \in [0, 1].$$

Then each $(T, I, F) \in \mathcal{T}^3$ effectively behaves like a real triple in $[0, 1]^3$. Define

$$\widetilde{S}(x) = \Big\{(\langle T, T, T \rangle, \langle I, I, I \rangle, \langle F, F, F \rangle) \;\Big|\; (T, I, F) \in \tilde{A}(x)\Big\}.$$

Hence, if $\tilde{A}(x) \subseteq [0, 1]^3$, we have a corresponding set of triangular triplets degenerating to points. Thus, every real-based HNS can be viewed as a degenerate special case of an SVTHNS. □

Definition 2.12 (Single-Valued Triangular n-SuperHyperneutrosophic Set (SVT-n-SHNS))**.** Let X be a non-empty set, and define recursively:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)) \quad \text{for } k \ge 2.$$

Likewise, let $\mathcal{T}$ be the set of all triangular fuzzy numbers on the real line, and consider $\mathcal{P}_n(\mathcal{T}^3)$ for n-nested subsets of $\mathcal{T}^3$.

A *Single-Valued Triangular n-SuperHyperneutrosophic Set (SVT-n-SHNS)* is a mapping

$$\widetilde{S}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n(\mathcal{T}^3),$$

such that for each $A \in \mathcal{P}_n(X)$, $\widetilde{S}_n(A) \subseteq \mathcal{T}^3$ (or a set of such triplets), each triplet being $\langle T, I, F \rangle \in \mathcal{T}^3$ in the triangular fuzzy sense.

Theorem 2.13. *Every Single-Valued Triangular Hyperneutrosophic Set is a special case of a Single-Valued Triangular n-SuperHyperneutrosophic Set (SVT-n-SHNS), obtained by letting $n = 1$.*

Proof. A *Single-Valued Triangular Hyperneutrosophic Set (SVTHNS)* $\widetilde{S}$ has $\widetilde{S}(x) \subseteq \mathcal{T}^3$. In Definition 2.12, let $n = 1$. Then

$$\widetilde{S}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \;\to\; \mathcal{P}_1(\mathcal{T}^3) = \mathcal{P}(\mathcal{T}^3).$$

We can define

$$\widetilde{S}_1(\{x\}) := \widetilde{S}(x), \quad \widetilde{S}_1(A) = \varnothing \text{ for } A \neq \{x\}.$$

Hence, restricting to singletons $\{x\} \subseteq X$ recovers precisely the membership sets from $\widetilde{S}(x)$. Therefore, every SVTHNS is subsumed under an SVT-1-SHNS scenario. □

Theorem 2.14. *Every n-SuperHyperneutrosophic Set is a special case of a Single-Valued Triangular n-SuperHyperneutrosophic Set, by making each triangular membership degenerate to a single real value.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ satisfies

$$\tilde{A}_n(A) \;\subseteq\; [0,1]^3,$$

or some similar real domain for each $A \in \mathcal{P}_n(X)$. Compare with Definition 2.12, where $\widetilde{S}_n(A) \subseteq \mathcal{T}^3$. If we let each triangular fuzzy number $\langle \alpha_l, \alpha_m, \alpha_r \rangle$ collapse to $\langle r, r, r \rangle \in [0,1]$ for some real $r \in [0,1]$, then

$$\langle T_l, T_m, T_r \rangle = \langle T, T, T \rangle, \quad \langle I_l, I_m, I_r \rangle = \langle I, I, I \rangle, \quad \langle F_l, F_m, F_r \rangle = \langle F, F, F \rangle.$$

Thus, define

$$\widetilde{S}_n(A) = \Big\{ \big(\langle T,T,T \rangle, \langle I,I,I \rangle, \langle F,F,F \rangle \big) \;\Big|\; (T,I,F) \in \tilde{A}_n(A) \Big\}.$$

This exactly reproduces the real triplet approach in the n-SuperHyperneutrosophic Set. Hence, n-SHNS emerges as a degenerate special case of SVT-n-SHNS. □

2.3 Fermatean Neutrosophic Set

A Fermatean Neutrosophic Set represents truth, indeterminacy, and falsity memberships satisfying $(T_F)^3 + (I_F)^3 + (F_F)^3 \leq 2$ [3, 15–17, 21, 50, 61]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.15 (Fermatean Neutrosophic Set)**.** (cf. [16, 50, 61]) A *Fermatean Neutrosophic Set (FNS)* on a universe A is defined as:

$$F = \{ \langle x, T_F(x), I_F(x), F_F(x) \rangle \mid x \in A \},$$

where the following conditions hold:

1. $T_F(x), I_F(x), F_F(x) \in [0,1]$ for all $x \in A$,
2. $(T_F(x))^3 + (I_F(x))^3 + (F_F(x))^3 \leq 2$.

Here:

- $T_F(x)$: The degree of truth of the element x to the set F,
- $I_F(x)$: The degree of indeterminacy of x to the set F,
- $F_F(x)$: The degree of falsity of x to the set F.

Definition 2.16 (Fermatean Hyperneutrosophic Set (FHNS))**.** Let X be a non-empty set, and let

$$\mathcal{F} \;\subseteq\; \{(T,I,F) \in [0,1]^3 : T^3 + I^3 + F^3 \;\leq\; 2\}.$$

A *Fermatean Hyperneutrosophic Set (FHNS)* $\widetilde{F}$ on X is a mapping

$$\widetilde{F} : X \;\to\; \mathcal{P}(\mathcal{F}),$$

where for each $x \in X$,

$$\widetilde{F}(x) \;\subseteq\; \big\{ (T,I,F) \in [0,1]^3 : T^3 + I^3 + F^3 \;\leq\; 2 \big\}.$$

In simpler terms, each $x \in X$ is assigned a *set* of Fermatean triplets (T, I, F), each triplet satisfying

$$T^3 + I^3 + F^3 \;\leq\; 2, \quad (T,I,F) \in [0,1]^3.$$

Theorem 2.17. *Every Fermatean Neutrosophic Set is a special case of a Fermatean Hyperneutrosophic Set.*

Proof. A *Fermatean Neutrosophic Set (FNS) F* on A assigns each $x \in A$ a single triplet $\big(T_F(x), I_F(x), F_F(x)\big) \in [0,1]^3$ satisfying $\big(T_F(x)\big)^3 + \big(I_F(x)\big)^3 + \big(F_F(x)\big)^3 \leq 2$. By Definition 2.16, we allow each x a *set* of such triplets. Let

$$\widetilde{F}(x) = \Big\{\big(T_F(x), I_F(x), F_F(x)\big)\Big\},$$

a singleton set. Then the same cubic constraint holds, so each single-valued FNS is embedded in the FHNS framework as a degenerate (singleton) case. □

Theorem 2.18. *Every (standard) Hyperneutrosophic Set is a special case of a Fermatean Hyperneutrosophic Set when the cubic constraint is omitted or replaced by the usual sum constraint.*

Proof. A standard *Hyperneutrosophic Set (HNS) $\tilde{A}$* has

$$\tilde{A}(x) \subseteq [0,1]^3 \quad \text{with sum constraints e.g. } T + I + F \leq 3.$$

In Definition 2.16, we require $T^3 + I^3 + F^3 \leq 2$. Observe that if we *ignore* or *do not enforce* the Fermatean condition, or equivalently treat the entire region $[0,1]^3$ as permissible, we obtain a typical HNS definition. Formally, define

$$\widetilde{F}(x) = \tilde{A}(x) \quad \text{and treat the Fermatean constraint as optional.}$$

Hence the real hyperneutrosophic membership sets appear as a sub-case within FHNS by ignoring the cubic bound. Therefore, a standard HNS emerges as a special scenario of FHNS lacking the Fermatean constraint. □

Definition 2.19 (Fermatean n-SuperHyperneutrosophic Set (F-n-SHNS))**.** Let X be a non-empty set. Define recursively:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad \text{for } k \geq 2.$$

Let

$$\mathcal{F} = \big\{(T, I, F) \in [0,1]^3 : T^3 + I^3 + F^3 \leq 2\big\}.$$

Then the *Fermatean n-SuperHyperneutrosophic Set (F-n-SHNS)* is a mapping

$$\widetilde{F}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n(\mathcal{F}),$$

meaning that for any $A \in \mathcal{P}_n(X)$,

$$\widetilde{F}_n(A) \subseteq \mathcal{F} = \big\{(T, I, F) \in [0,1]^3 : T^3 + I^3 + F^3 \leq 2\big\}.$$

Thus, each n-th level subset $A \subseteq X$ is mapped to a *set* of Fermatean membership triplets, all satisfying $T^3 + I^3 + F^3 \leq 2$.

Theorem 2.20. *Every Fermatean Hyperneutrosophic Set is a special case of a Fermatean n-SuperHyperneutrosophic Set (F-n-SHNS), by setting $n = 1$.*

Proof. A *Fermatean Hyperneutrosophic Set (FHNS) $\widetilde{F}$* has $\widetilde{F}(x) \subseteq \{(T, I, F) \in [0,1]^3 : T^3 + I^3 + F^3 \leq 2\}$. In Definition 2.19, let $n = 1$, so $\mathcal{P}_1(X) = \mathcal{P}(X)$. Define

$$\widetilde{F}_1(\{x\}) := \widetilde{F}(x), \quad \widetilde{F}_1(A) = \varnothing \text{ for } A \neq \{x\}.$$

Hence each singleton $\{x\} \subseteq X$ recovers precisely the membership sets from the original FHNS. The constraint $T^3 + I^3 + F^3 \leq 2$ remains identical. Consequently, we embed FHNS as a special single-level version of F-n-SHNS. □

Theorem 2.21. *Every (standard) n-SuperHyperneutrosophic Set is a special case of a Fermatean n-SuperHyperneutrosophic Set by ignoring the Fermatean cubic constraint.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ satisfies

$$\tilde{A}_n(A) \subseteq [0,1]^3$$

for each $A \in \mathcal{P}_n(X)$. In Definition 2.19, we require $(T, I, F) \in [0,1]^3$ with $T^3 + I^3 + F^3 \leq 2$. If we do not impose that extra condition (or treat it as automatically satisfied for all $(T, I, F) \in [0,1]^3$), we revert to an n-SHNS. Formally, define

$$\widetilde{F}_n(A) = \tilde{A}_n(A),$$

and skip the cubic constraint. Thus, the usual n-SuperHyperneutrosophic membership sets appear as a subcase of the Fermatean approach. Hence, every n-SHNS is included in F-n-SHNS by discarding the Fermatean requirement. □

2.4 Linguistic HyperNeutrosophic Set

A Linguistic Neutrosophic Set integrates linguistic terms with truth, indeterminacy, and falsity degrees, addressing uncertainty and vagueness in linguistic assessments [18, 19, 52, 73–76, 82]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.22. (cf. [18,52,75]) Let X be a non-empty set (universe of discourse) and $S = \{s_i \mid i = 1, 2, \ldots, t\}$ a linguistic term set, where each s_i represents a linguistic term (e.g., "low", "medium", "high"). A *Linguistic Neutrosophic Set (LNS)* is defined as:

$$\mathcal{S} = \{\langle s_x, T(s_x), I(s_x), F(s_x)\rangle \mid s_x \in S\},$$

where $T(s_x), I(s_x), F(s_x)$ are linguistic truth-membership, indeterminacy-membership, and falsity-membership values, respectively, for the term s_x. These are linguistic variables representing the degrees of truth, indeterminacy, and falsity associated with s_x.

Each $T(s_x), I(s_x), F(s_x)$ satisfies the following properties:

$$T(s_x), I(s_x), F(s_x) \in [0,1], \quad \forall s_x \in S.$$

Furthermore, the operational laws for linguistic neutrosophic numbers (LNNs) $l_1 = \langle T_1, I_1, F_1\rangle$ and $l_2 = \langle T_2, I_2, F_2\rangle$ are defined as:

$$\begin{aligned}
l_1 \oplus l_2 &= \langle T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2\rangle,\\
l_1 \ominus l_2 &= \langle \frac{T_1 - T_2}{1 - T_2}, \frac{I_1}{I_2}, \frac{F_1}{F_2}\rangle,\\
l_1 \otimes l_2 &= \langle T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2\rangle,\\
l_1 \oslash l_2 &= \langle \frac{T_1}{T_2}, \frac{I_1 - I_2}{1 - I_2}, \frac{F_1 - F_2}{1 - F_2}\rangle,\\
\lambda l_1 &= \langle 1 - (1 - T_1)^\lambda, I_1^\lambda, F_1^\lambda\rangle,\\
l_1^\lambda &= \langle T_1^\lambda, 1 - (1 - I_1)^\lambda, 1 - (1 - F_1)^\lambda\rangle.
\end{aligned}$$

Definition 2.23 (Linguistic Hyperneutrosophic Set (LHNS))**.** Let $S = \{s_1, s_2, \ldots, s_t\}$ be a linguistic term set, and let $\mathcal{L} \subseteq [0,1]^3$ represent the domain of possible *linguistic neutrosophic numbers* (LNNs), subject to

$$T(s_x),\ I(s_x),\ F(s_x) \in [0,1] \quad \text{for each } s_x \in S.$$

A *Linguistic Hyperneutrosophic Set (LHNS)* is a mapping

$$\widetilde{L} : S \ \to\ \mathcal{P}(\mathcal{L}),$$

where each $\widetilde{L}(s_x) \subseteq \mathcal{L}$. Concretely, for each linguistic term $s_x \in S$,

$$\widetilde{L}(s_x) \ \subseteq\ \big\{(T, I, F) \in [0,1]^3 :\ \text{each } (T, I, F) \text{ can be interpreted as a linguistic neutrosophic number}\big\}.$$

Hence, each linguistic term s_x is assigned a *set* of LNN-based triplets (T, I, F), capturing diverse or uncertain opinions regarding truth, indeterminacy, and falsity degrees.

Theorem 2.24. *Every Linguistic Neutrosophic Set (LNS) is a special case of a Linguistic Hyperneutrosophic Set (LHNS).*

Proof. A *Linguistic Neutrosophic Set (LNS)* $\mathcal{S}$ assigns exactly one linguistic neutrosophic number $(T(s_x), I(s_x), F(s_x))$ to each $s_x \in S$. In Definition 2.23, we allow each s_x a *set* of such triplets. Let

$$\widetilde{L}(s_x) = \Big\{\big(T(s_x), I(s_x), F(s_x)\big)\Big\},$$

i.e., a singleton set. Hence the single-valued LNS membership is realized in the LHNS context as a singleton for each linguistic term. Consequently, every LNS is a degenerate (singleton) LHNS. □

Theorem 2.25. *Every standard Hyperneutrosophic Set is a special case of a Linguistic Hyperneutrosophic Set by removing the linguistic layer and focusing on numeric real values in* $[0, 1]$.

Proof. A standard *Hyperneutrosophic Set (HNS)* $\tilde{A}$ is typically a mapping from a non-empty set X (or S) to subsets of $[0, 1]^3$ with constraints on (T, I, F). In Definition 2.23, we define each membership set in $\mathcal{L} \subseteq [0, 1]^3$ with a linguistic interpretation. If we ignore or do not require the linguistic semantics and treat (T, I, F) purely as real-based memberships (i.e., ignoring the LNN operations or labels), we recover the same structure as HNS. Formally, define

$$\widetilde{L}(s_x) = \tilde{A}(s_x),$$

and let the linguistic viewpoint be optional. This reproduces the real hyperneutrosophic membership sets. Thus, ignoring the linguistic dimension reverts LHNS to a standard HNS. □

Definition 2.26 (Linguistic n-SuperHyperneutrosophic Set (L-n-SHNS))**.** Let $S = \{s_1, s_2, \dots, s_t\}$ be a linguistic term set, and let $\mathcal{L} \subseteq [0, 1]^3$ denote the domain of possible LNN-based triplets (T, I, F). Define:

$$\mathcal{P}_1(S) = \mathcal{P}(S), \quad \mathcal{P}_k(S) = \mathcal{P}\big(\mathcal{P}_{k-1}(S)\big) \quad \text{for } k \geq 2.$$

Likewise, consider $\mathcal{P}_n(\mathcal{L})$, the n-th nested family of subsets of $\mathcal{L}$.

A *Linguistic n-SuperHyperneutrosophic Set (L-n-SHNS)* is a mapping

$$\widetilde{L}_n : \mathcal{P}_n(S) \longrightarrow \mathcal{P}_n(\mathcal{L}),$$

meaning that for each $A \in \mathcal{P}_n(S)$, $\widetilde{L}_n(A) \subseteq \mathcal{L}$ is a *set* of LNN triplets in $[0, 1]^3$, capturing the linguistic truth, indeterminacy, and falsity degrees for the n-th level subset A.

Theorem 2.27. *Every Linguistic Hyperneutrosophic Set is a special case of a Linguistic n-SuperHyperneutrosophic Set (L-n-SHNS) for* $n = 1$.

Proof. A *Linguistic Hyperneutrosophic Set (LHNS)* $\widetilde{L}$ has $\widetilde{L}(s_x) \subseteq \mathcal{L} \subseteq [0, 1]^3$ for each $s_x \in S$. In Definition 2.26, let $n = 1$. Then

$$\widetilde{L}_1 : \mathcal{P}_1(S) = \mathcal{P}(S) \longrightarrow \mathcal{P}_1(\mathcal{L}) = \mathcal{P}(\mathcal{L}).$$

For each singleton $\{s_x\} \subseteq S$, define

$$\widetilde{L}_1(\{s_x\}) := \widetilde{L}(s_x), \quad \widetilde{L}_1(A) = \varnothing \text{ for } A \neq \{s_x\}.$$

Hence, restricting to singletons recovers the LHNS membership sets. Thus, any LHNS is embedded in an L-1-SHNS as a single-level scenario. □

Theorem 2.28. *Every standard n-SuperHyperneutrosophic Set is a special case of a Linguistic n-SuperHyperneutrosophic Set, by disregarding the linguistic interpretation and using real values in* $[0, 1]^3$.

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ is a mapping from $\mathcal{P}_n(U)$ (for some universe U) to subsets of $[0, 1]^3$. In Definition 2.26, we map $\mathcal{P}_n(S)$ to subsets of $\mathcal{L} \subseteq [0, 1]^3$, with a linguistic dimension. If we ignore the linguistic labeling (or treat s_x as just an element in U) and interpret (T, I, F) purely as real degrees in $[0, 1]^3$, the structure coincides with a standard n-SHNS. Formally:

$$\widetilde{L}_n(A) = \tilde{A}_n(A), \quad \text{and ignore linguistic semantics.}$$

Hence, each membership set in $[0, 1]^3$ appears identically in the L-n-SHNS. Therefore, an n-SHNS emerges as a sub-case of L-n-SHNS by skipping the linguistic layer. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Samah Ibrahim Abdel Aal, Mahmoud MA Abd Ellatif, and Mohamed Monir Hassan. *Proposed model for evaluating information systems quality based on single valued triangular neutrosophic numbers*. Infinite Study, 2018.

[2] Samah Ibrahim Abdel Aal, Mahmoud MA Abd Ellatif, and Mohamed Monir Hassan. *Two ranking methods of single valued triangular neutrosophic numbers to rank and evaluate information systems quality*. Infinite Study, 2018.

[3] Abdulkhaleq Abdulkhaleq. Interval-valued fermatean neutrosophic graph with grey wolf optimization for sarcasm recognition on microblogging data. *International Journal of Neutrosophic Science*, 2024.

[4] Mumtaz Ali, Luu Quoc Dat, Le Hoang Son, and Florentin Smarandache. Interval complex neutrosophic set: Formulation and applications in decision-making. *International Journal of Fuzzy Systems*, 20:986 – 999, 2017.

[5] Mumtaz Ali and Florentin Smarandache. Complex neutrosophic set. *Neural Computing and Applications*, 28:1817–1834, 2016.

[6] Mohammed Alqahtani, Murugan Kaviyarasu, Anas Al-Masarwah, and Murugesan Rajeshwari. Application of complex neutrosophic graphs in hospital infrastructure design. *Mathematics*, 2024.

[7] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[8] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[9] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[10] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[11] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[12] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[13] Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. Complex neutrosophic graphs of type 1. *viXra*, 2017.

[14] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and V. Venkateswara Rao. Bipolar complex neutrosophic graphs of type 1. *viXra*, 2018.

[15] Said Broumi, S Krishna Prabha, and Vakkas Uluçay. Interval-valued fermatean neutrosophic shortest path problem via score function. *Neutrosophic Systems with Applications*, 11:1–10, 2023.

[16] Said Broumi, Swaminathan Mohanaselvi, Tomasz Witczak, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Complex fermatean neutrosophic graph and application to decision making. *Decision Making: Applications in Management and Engineering*, 2023.

[17] Said Broumi, R Sundareswaran, M Shanmugapriya, Assia Bakali, and Mohamed Talea. Theory and applications of fermatean neutrosophic graphs. *Neutrosophic sets and systems*, 50:248–286, 2022.

[18] Huakun Chen, Jingping Shi, Yongxi Lyu, and Qianlei Jia. A decision-making model with cloud model, z-numbers, and interval-valued linguistic neutrosophic sets. *Entropy*, 2024.

[19] Huakun Chen, Jingping Shi, Yongxi Lyu, and Qianlei Jia. A decision-making model with cloud model, z-numbers, and interval-valued linguistic neutrosophic sets. *Entropy*, 26(11):892, 2024.

[20] S Dhouib. Optimization of travelling salesman problem on single valued triangular neutrosophic number using dhouib-matrix-tsp1 heuristic. *International Journal of Engineering*, 34(12):2642–2647, 2021.

[21] Souhail Dhouib, Said Broumi, Mohamed Talea, et al. Solving the minimum spanning tree problem under interval-valued fermatean neutrosophic domain. *Neutrosophic Sets and Systems*, 67(1):2, 2024.

[22] Jianping Fan, Xuefei Jia, and Meiqin Wu. Green supplier selection based on dombi prioritized bonferroni mean operator with single-valued triangular neutrosophic sets. *International Journal of Computational Intelligence Systems*, 12(2):1091–1101, 2019.

[23] Jianping Fan, Xuefei Jia, and Meiqin Wu. A new multi-criteria group decision model based on single-valued triangular neutrosophic sets and edas method. *Journal of Intelligent & Fuzzy Systems*, 38(2):2089–2102, 2020.

[24] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[25] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[26] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[27] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[28] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[29] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[30] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[31] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[32] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[33] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[34] Takaaki Fujita. Short note of even-hole-graph for uncertain graph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 351, 2025.

[35] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[36] Takaaki Fujita and Florentin Smarandache. A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 308.

[37] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[38] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[39] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[40] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[41] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[42] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[43] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[44] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[45] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[46] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[47] Vinod Jangid and Ganesh Kumar. Matrix games with single-valued triangular neutrosophic numbers as pay-offs. *Neutrosophic Sets and Systems*, 45(1):14, 2021.

[48] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[49] Gözde Koca, Ezgi Demir, Özgür İcan, and Çağlar Karamaşa. Analyzing shortest path problem via single-valued triangular neutrosophic numbers: A case study. *Neutrosophic Operational Research: Methods and Applications*, pages 559–573, 2021.

[50] S krishna Prabha, Said Broumi, Souhail Dhouib, and Mohamed Talea. Implementation of circle-breaking algorithm on fermatean neutrosophic graph to discover shortest path. *Neutrosophic Sets and Systems*, 72:256–271, 2024.

[51] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[52] Ying-Ying Li, Hongyu Zhang, and Jian-Qiang Wang. Linguistic neutrosophic sets and their application in multicriteria decision-making problems. *International Journal for Uncertainty Quantification*, 7(2), 2017.

[53] Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: New social network models. *Algorithms*, 12:234, 2019.

[54] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[55] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[56] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[57] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[58] Zdzis law Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[59] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[60] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[61] P Roopadevi, M Karpagadevi, S Krishnaprakash, Said Broumi, and S Gomathi. Comprehensive decision-making with spherical fermatean neutrosophic sets in structural engineering. *International Journal of Neutrosophic Science (IJNS)*, 24(4), 2024.

[62] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[63] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[64] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[65] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[66] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[67] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[68] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[69] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[70] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[71] Subadhra Srinivas and K Prabakaran. Optimization of single-valued triangular neutrosophic fuzzy travelling salesman problem. *Neutrosophic Sets and Systems*, 60(1):26, 2023.

[72] P. Thirunavukarasu and R. Suresh. On regular complex neutrosophic graphs. 2017.

[73] Li Wang. Linguistic neutrosophic sets with application to group decision-making to enhance the work effectiveness evaluation of university counselors. *Neutrosophic Sets and Systems*, 77:129–152, 2025.

[74] Nunu Wang and Hongyu Zhang. Probability multivalued linguistic neutrosophic sets for multi-criteria group decision-making. *International Journal for Uncertainty Quantification*, 7(3), 2017.

[75] Xingang Wang, Yushui Geng, Peipei Yao, and Mengjie Yang. Multiple attribute group decision making approach based on extended vikor and linguistic neutrosophic set. *Journal of Intelligent & Fuzzy Systems*, 36(1):149–160, 2019.

[76] Jun Ye, Shigui Du, and Rui Yong. Mine safety evaluation method using correlation coefficients of consistency linguistic neutrosophic sets in a linguistic neutrosophic multivalued environment. *Soft Computing*, 27(13):8599–8609, 2023.

[77] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[78] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[79] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[80] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[81] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[82] Mengwei Zhao, Guiwu Wei, Jiang Wu, Yanfeng Guo, and Cun Wei. Todim method for multiple attribute group decision making based on cumulative prospect theory with 2-tuple linguistic neutrosophic sets. *International Journal of Intelligent Systems*, 36(3):1199–1222, 2021.

Chapter 5
Some Types of HyperNeutrosophic Set (3): Dynamic, Quadripartitioned, Pentapartitioned, Heptapartitioned, m-polar

Takaaki Fujita[1]*, *Florentin Smarandache*[2],
[1]* *Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp*
[2] *University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu*

Abstract

This paper builds upon the foundation established in [50, 51]. The Neutrosophic Set provides a robust mathematical framework for handling uncertainty, defined by three membership functions: truth, indeterminacy, and falsity. Recent developments have introduced extensions such as the Hyperneutrosophic Set and SuperHyperneutrosophic Set to tackle increasingly complex and multidimensional problems.

In this study, we explore further extensions, including the Dynamic Neutrosophic Set, Quadripartitioned Neutrosophic Set, Pentapartitioned Neutrosophic Set, Heptapartitioned Neutrosophic Set, and m-Polar Neutrosophic Set, to address advanced challenges and applications.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section introduces the key concepts and definitions fundamental to the discussions in this paper. The study employs standard set-theoretic operations and extends them to advanced frameworks. Readers seeking an in-depth understanding of classical set theory are referred to resources such as [25, 68, 71, 72]. For foundational principles and applications of Neutrosophic Sets, the referenced literature provides comprehensive insights.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

Various set-theoretic frameworks have been devised to address uncertainty, vagueness, and imprecision in decision-making. These frameworks include Fuzzy Sets [120–124], Intuitionistic Fuzzy Sets [6–11], Vague Sets [2, 12, 64, 73, 97], Plithogenic Sets [27, 34, 37–39, 47, 48, 60, 109, 111, 112], Soft Sets [65, 74, 79], Hypersoft Sets [30, 42, 59, 110], and Neutrosophic Sets [31, 32, 35, 46, 52–57, 61, 62, 106, 107, 114].

Neutrosophic Sets extend the concept of Fuzzy Sets by incorporating a third dimension—indeterminacy—alongside truth and falsity [104–107]. This approach enables a more nuanced representation of uncertainty and ambiguity. Further advancements have resulted in the development of HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, which are tailored for addressing more intricate and high-dimensional problems [29, 41].

The following subsections present precise definitions and critical attributes of these extended frameworks.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [37, 94] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [26, 37, 43, 103, 113])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Neutrosophic Set)**.** [106, 107] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 1.5 (HyperNeutrosophic Set)**.** (cf. [29, 41, 44, 45, 108]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Definition 1.6 (n-SuperHyperNeutrosophic Set)**.** (cf. [29, 41, 44, 45]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Dymanic HyperNeutrosophic set

A Dynamic Neutrosophic Set incorporates time-dependent truth, indeterminacy, and falsity functions, evolving continuously within a time domain [19, 84, 96, 115–117, 119]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.1 (Dynamic Neutrosophic Set)**.** (cf. [116]) Let U be a universal set, and $A \subseteq U$ be a neutrosophic set. A *Dynamic Neutrosophic Set (DNS)* is defined with respect to a time parameter t (where $t \in T, T \subseteq \mathbb{R}^+$) as:

$$D_A^t = \{\langle x, T_A(t), I_A(t), F_A(t)\rangle \mid x \in U\},$$

where:

- $T_A(t) : U \to [0, 1]$ is the *truth-membership function* at time t,
- $I_A(t) : U \to [0, 1]$ is the *indeterminacy-membership function* at time t,
- $F_A(t) : U \to [0, 1]$ is the *falsity-membership function* at time t,

such that for all $x \in U$,

$$T_A(t) + I_A(t) + F_A(t) \leq 1.$$

The functions $T_A(t), I_A(t), F_A(t)$ are time-dependent and continuous over T. The evolution of the neutrosophic components is represented as:

$$T_A(t), I_A(t), F_A(t) : T \to [0, 1].$$

Definition 2.2 (Dynamic Hyperneutrosophic Set (DHNS))**.** Let X be a non-empty set, and let $T \subseteq \mathbb{R}^+$ be a time domain. A *Dynamic Hyperneutrosophic Set* (DHNS) $\widetilde{D}$ on X is specified by a mapping:

$$\widetilde{D} : X \times T \quad \to \quad \mathcal{P}([0, 1]^3),$$

such that for each $(x, t) \in X \times T$,

$$\widetilde{D}(x, t) \;\subseteq\; \{(T, I, F) \in [0, 1]^3 : T + I + F \leq 3\}.$$

Equivalently, for each $t \in T$, the function

$$\widetilde{D}_t : X \to \mathcal{P}([0, 1]^3), \quad \widetilde{D}_t(x) := \widetilde{D}(x, t),$$

is a Hyperneutrosophic Set in the usual sense, but one that evolves over time t. Each $\widetilde{D}_t$ is presumably continuous in t in some sense (optional constraint), reflecting how membership sets might change as t progresses.

Theorem 2.3. *Every Dynamic Neutrosophic Set is a special case of a Dynamic Hyperneutrosophic Set.*

Proof. A *Dynamic Neutrosophic Set (DNS)* $\mathcal{D}$ on U (Definition **??**) assigns each (x, t) a single triplet $(T_A(t)(x), I_A(t)(x), F_A(t)(x))$. Let

$$\widetilde{D}(x, t) = \Big\{(T_A(t)(x),\ I_A(t)(x),\ F_A(t)(x))\Big\}.$$

Thus, each (x, t) maps to a *singleton* in $[0, 1]^3$. We see that $\widetilde{D} : X \times T \to \mathcal{P}([0, 1]^3)$ is a Dynamic Hyperneutrosophic Set. The only difference is that $\widetilde{D}(x, t)$ remains single-valued. Hence, every DNS is embedded in DHNS as a degenerate (singleton) membership set for each (x, t). □

Theorem 2.4. *Every Hyperneutrosophic Set is a special case of a Dynamic Hyperneutrosophic Set by ignoring time dependence or taking T to be a singleton domain.*

Proof. A *Hyperneutrosophic Set (HNS)* $\tilde{A}$ is a mapping $X \to \mathcal{P}([0, 1]^3)$. In Definition 2.2, we have $\widetilde{D} : X \times T \to \mathcal{P}([0, 1]^3)$. If we fix $t = t_0$ or let $T = \{t_0\}$ be a single point in time, then

$$\widetilde{D}(x, t_0) = \tilde{A}(x).$$

Hence, ignoring or collapsing the time axis recovers the standard hyperneutrosophic membership. Thus, HNS is a sub-case of DHNS where time is trivial or absent. □

Definition 2.5 (Dynamic n-SuperHyperneutrosophic Set (D-n-SHNS))**.** Let X be a non-empty set and $T \subseteq \mathbb{R}^+$ be the time domain. Define

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \geq 2).$$

Similarly, let $\mathcal{P}_n\big([0,1]^3\big)$ represent the n-th nested family of non-empty subsets of $[0,1]^3$. A *Dynamic n-SuperHyperneutrosophic Set (D-n-SHNS)* is a mapping:

$$\widetilde{D}_n : \mathcal{P}_n(X) \times T \longrightarrow \mathcal{P}_n\big([0,1]^3\big),$$

such that for each $(A,t) \in \mathcal{P}_n(X) \times T$:

$$\widetilde{D}_n(A,t) \subseteq \big\{(T,I,F) \in [0,1]^3 : T + I + F \leq 3\big\}.$$

That is, each n-th level subset A is assigned a *set* of membership triplets (T,I,F), each lying in $[0,1]^3$ with $T + I + F \leq 3$, and these sets vary over the time parameter t.

Theorem 2.6. *Every Dynamic Hyperneutrosophic Set is a special case of a Dynamic n-SuperHyperneutrosophic Set (D-n-SHNS) for $n = 1$.*

Proof. A *Dynamic Hyperneutrosophic Set (DHNS)* $\widetilde{D}$ is a mapping $X \times T \to \mathcal{P}([0,1]^3)$. In Definition 2.5, set $n = 1$. Then $\mathcal{P}_1(X) = \mathcal{P}(X)$, and we define

$$\widetilde{D}_1(A,t) = \begin{cases} \widetilde{D}(x,t), & \text{if } A = \{x\}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Hence, for singletons $A = \{x\}$, we recover exactly the membership sets from $\widetilde{D}(x,t)$. Therefore, a DHNS is embedded in D-1-SHNS as a single-level case. □

Theorem 2.7. *Every (static) n-SuperHyperneutrosophic Set is a special case of a Dynamic n-SuperHyperneutrosophic Set by taking T to be a single point or ignoring time.*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ is a mapping $\mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3)$. In Definition 2.5, we have $\widetilde{D}_n : \mathcal{P}_n(X) \times T \to \mathcal{P}_n([0,1]^3)$. If we let $T = \{t_0\}$ be a single point in time (or otherwise disregard time), we can define

$$\widetilde{D}_n(A,t_0) = \tilde{A}_n(A).$$

Then ignoring the time dimension yields the standard n-SuperHyperneutrosophic membership sets. Thus, every n-SHNS is included in D-n-SHNS by collapsing T to a singleton or skipping time dependence. □

2.2 Hyper Quadripartitioned Neutrosophic set

A Quadripartitioned Neutrosophic Set assigns four membership values [16,17,23,49,69,70,86,88,99–101,104]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.8 (Quadripartitioned Neutrosophic Set (QNS))**.** (cf. [16,23,69]) Let X be a universe of discourse. A *Quadripartitioned Neutrosophic Set (QNS)* on X is given by

$$QNS = \{\langle x, T(x), C(x), U(x), F(x)\rangle \mid x \in X\},$$

where each of $T(x), C(x), U(x), F(x)$ lies in $[0,1]$, satisfying

$$0 \leq T(x) + C(x) + U(x) + F(x) \leq 4.$$

Definition 2.9 (Hyper Quadripartitioned Neutrosophic Set (HQNS))**.** Let X be a non-empty set, and consider the family $\mathcal{P}([0,1]^4)$ of all non-empty subsets of the unit 4-cube $[0,1]^4$. A *Hyper Quadripartitioned Neutrosophic Set (HQNS)* $\widetilde{Q}$ on X is a mapping

$$\widetilde{Q} : X \longrightarrow \mathcal{P}([0,1]^4),$$

such that for each $x \in X$,

$$\widetilde{Q}(x) \subseteq \Big\{(T,C,U,F) \in [0,1]^4 : T + C + U + F \leq 4\Big\}.$$

That is, each point $x \in X$ is assigned a *set* of quadripartitioned membership quadruples $\big(T,C,U,F\big)$, each lying in $[0,1]^4$ with $T + C + U + F \leq 4$.

Theorem 2.10. *Every Quadripartitioned Neutrosophic Set is a special case of a Hyper Quadripartitioned Neutrosophic Set.*

Proof. A *Quadripartitioned Neutrosophic Set (QNS) Q* assigns each $x \in X$ exactly one quadruple

$$(T(x), C(x), U(x), F(x)) \in [0,1]^4$$

with $T + C + U + F \leq 4$. In Definition 2.9, we let

$$\widetilde{Q}(x) = \Big\{\big(T(x),\ C(x),\ U(x),\ F(x)\big)\Big\},$$

a *singleton* subset of $[0,1]^4$. The same constraint $T + C + U + F \leq 4$ persists, so each single-valued QNS is embedded in the HQNS framework as a degenerate (singleton) membership set for each x. □

Theorem 2.11. *Every HyperNeutrosophic Set is a special case of a Hyper Quadripartitioned Neutrosophic Set by treating the four components as, for instance, $(T, I, 0, F)$ (or by ignoring C and U).*

Proof. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* maps each $x \in X$ to a subset of $[0,1]^3$ with $T + I + F \leq 3$. In Definition 2.9, each membership is a subset of $[0,1]^4$ with $T + C + U + F \leq 4$. If we fix $C = 0$ and $U = I$ (or $U = 0$) so that effectively (T, C, U, F) becomes $(T, 0, I, F)$, and require $T + 0 + I + F = T + I + F \leq 3$, we see that ignoring or collapsing the extra dimension recovers an HNS membership set. Specifically, define

$$\widetilde{Q}(x) = \Big\{\big(T, I, 0, F\big) : (T, I, F) \in \tilde{A}(x)\Big\}.$$

Hence, ignoring two of the four components (or setting them to zero or merging them) yields the usual 3-component hyperneutrosophic membership. Therefore, an HNS is subsumed under HQNS by discarding or collapsing extra components to zero. □

Definition 2.12 (n-SuperHyper Quadripartitioned Neutrosophic Set (n-SHQNS))**.** Let X be a non-empty set. Define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \ \ (k \geq 2).$$

Similarly, let $\mathcal{P}_n\big([0,1]^4\big)$ denote the n-th nested family of non-empty subsets of $[0,1]^4$. A *n-SuperHyper Quadripartitioned Neutrosophic Set (n-SHQNS)* is a mapping

$$\widetilde{Q}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big([0,1]^4\big),$$

such that for each $A \in \mathcal{P}_n(X)$,

$$\widetilde{Q}_n(A) \ \subseteq \ \Big\{(T, C, U, F) \in [0,1]^4 : \ T + C + U + F \leq 4\Big\}.$$

Hence, each n-th level subset $A \subseteq X$ is assigned a *set* of four-part membership quadruples (T, C, U, F) in $[0,1]^4$ with $T + C + U + F \leq 4$.

Theorem 2.13. *Every Hyper Quadripartitioned Neutrosophic Set is a special case of an n-SuperHyper Quadripartitioned Neutrosophic Set (n-SHQNS) for $n = 1$.*

Proof. A *Hyper Quadripartitioned Neutrosophic Set (HQNS) $\widetilde{Q}$* is a mapping $X \to \mathcal{P}([0,1]^4)$ with $T + C + U + F \leq 4$. In Definition 2.12, set $n = 1$, so

$$\widetilde{Q}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \ \ \rightarrow \ \ \mathcal{P}_1\big([0,1]^4\big) = \mathcal{P}([0,1]^4).$$

Define

$$\widetilde{Q}_1(\{x\}) := \widetilde{Q}(x), \quad \widetilde{Q}_1(A) = \varnothing \quad \text{for } A \neq \{x\}.$$

Hence, for singletons $A = \{x\} \subseteq X$, we recover precisely $\widetilde{Q}(x)$. The constraint $T + C + U + F \leq 4$ remains the same, embedded in $\widetilde{Q}_1(A)$. Therefore, any HQNS is realized as a degenerate single-level mapping in an n-SHQNS with $n = 1$. □

Theorem 2.14. *Every n-SuperHyperNeutrosophic Set is a special case of an n-SuperHyper Quadripartitioned Neutrosophic Set by ignoring or collapsing the fourth component (e.g., $C = 0$ or $U = I$).*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ maps each $A \in \mathcal{P}_n(X)$ to a set of (T, I, F) in $[0,1]^3$ with $T + I + F \leq 3$. In Definition 2.12, each membership is in $[0,1]^4$ with $T + C + U + F \leq 4$. To retrieve an SHNS, fix or zero out some components. For instance, set $C = 0$, $U = I$, and require $T + I + F = T + (U) + F \leq 3$. Concretely, define

$$\widetilde{Q}_n(A) = \Big\{\big(T, 0, I, F\big) : (T, I, F) \in \tilde{A}_n(A)\Big\}.$$

Hence, ignoring the extra dimension(s) reverts membership to $(T, I, F) \in [0,1]^3$. Therefore, an n-SHNS is a special case of n-SHQNS with additional constraints or identification of dimensions. □

2.3 Hyper Pentapartitioned Neutrosophic set

A Pentapartitioned Neutrosophic Set assigns five membership values [5, 13, 15, 20–22, 28, 33, 36, 40, 58, 76, 85, 87]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.15 (Pentapartitioned Neutrosophic Set (PNS))**.** (cf. [13, 76, 85]) Let X be a universe of discourse. A *Pentapartitioned Neutrosophic Set (PNS)* on X is given by

$$PNS = \{\langle x, T(x), C(x), R(x), U(x), F(x)\rangle \mid x \in X\},$$

where each of $T(x), C(x), R(x), U(x), F(x) \in [0,1]$, satisfying

$$0 \;\leq\; T(x) + C(x) + R(x) + U(x) + F(x) \;\leq\; 5.$$

Definition 2.16 (Hyper Pentapartitioned Neutrosophic Set (HPNS))**.** Let X be a non-empty set, and let $\mathcal{P}([0,1]^5)$ denote the family of all non-empty subsets of the unit 5-cube $[0,1]^5$. A *Hyper Pentapartitioned Neutrosophic Set (HPNS)* $\widetilde{P}$ on X is a mapping

$$\widetilde{P} : X \;\to\; \mathcal{P}([0,1]^5),$$

such that for each $x \in X$,

$$\widetilde{P}(x) \;\subseteq\; \Big\{(T, C, R, U, F) \in [0,1]^5 : T + C + R + U + F \;\leq\; 5\Big\}.$$

Hence, each $x \in X$ is assigned a *set* of pentapartitioned membership quintuples $\big(T, C, R, U, F\big)$, where $T + C + R + U + F \leq 5$.

Theorem 2.17. *Every Pentapartitioned Neutrosophic Set is a special case of a Hyper Pentapartitioned Neutrosophic Set.*

Proof. A *Pentapartitioned Neutrosophic Set (PNS)* assigns each $x \in X$ a unique quintuple

$$(T(x), C(x), R(x), U(x), F(x)) \in [0,1]^5$$

satisfying $T + C + R + U + F \leq 5$. In Definition 2.16, we map each x to a *set* of such quintuples. Define:

$$\widetilde{P}(x) = \Big\{\big(T(x),\ C(x),\ R(x),\ U(x),\ F(x)\big)\Big\},$$

i.e. a singleton set. The same constraint $T + C + R + U + F \leq 5$ persists. Consequently, each PNS is embedded in HPNS as a degenerate (singleton) membership set for each x. □

Theorem 2.18. *Every HyperNeutrosophic Set is a special case of a Hyper Pentapartitioned Neutrosophic Set by reducing two membership components to zero (e.g., $C = R = 0$).*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps each $x \in X$ to a subset of $[0,1]^3$ (triplets (T, I, F) with $T + I + F \leq 3$). In Definition 2.16, each membership is a subset of $[0,1]^5$ with $T + C + R + U + F \leq 5$. If we identify (T, I, F) in $[0,1]^3$ with $(T, 0, 0, I, F)$ in $[0,1]^5$, we require $T + 0 + 0 + I + F = T + I + F \leq 3$. We can embed an HNS as:

$$\widetilde{P}(x) = \Big\{ (T, 0, 0, I, F) \,:\, (T, I, F) \in \tilde{A}(x) \Big\}.$$

Hence, ignoring or zeroing out two membership parts ($C = 0,\ R = 0$) recovers a 3-part hyperneutrosophic membership. Therefore, an HNS is included in HPNS by discarding extra membership dimensions. □

Definition 2.19 (n-SuperHyper Pentapartitioned Neutrosophic Set (n-SHPNS))**.** Let X be a non-empty set. Define recursively:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad \text{for } k \geq 2.$$

Similarly, let $\mathcal{P}_n([0,1]^5)$ denote the n-th nested family of non-empty subsets of $[0,1]^5$. A *n-SuperHyper Pentapartitioned Neutrosophic Set (n-SHPNS)* is a mapping

$$\widetilde{P}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big([0,1]^5\big),$$

such that for each $A \in \mathcal{P}_n(X)$,

$$\widetilde{P}_n(A) \subseteq \Big\{ (T, C, R, U, F) \in [0,1]^5 : \ T + C + R + U + F \leq 5 \Big\}.$$

Hence, each n-th level subset A is assigned a *set* of pentapartitioned membership quintuples $(T, C, R, U, F) \in [0,1]^5$, with $T + C + R + U + F \leq 5$.

Theorem 2.20. *Every Hyper Pentapartitioned Neutrosophic Set is a special case of an n-SuperHyper Pentapartitioned Neutrosophic Set (n-SHPNS) for $n = 1$.*

Proof. A *Hyper Pentapartitioned Neutrosophic Set (HPNS)* $\widetilde{P}$ is a mapping $X \to \mathcal{P}([0,1]^5)$ with $T + C + R + U + F \leq 5$. In Definition 2.19, set $n = 1$. Then

$$\widetilde{P}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \ \rightarrow \ \mathcal{P}_1\big([0,1]^5\big) = \mathcal{P}\big([0,1]^5\big).$$

Define

$$\widetilde{P}_1(\{x\}) := \widetilde{P}(x), \quad \widetilde{P}_1(A) = \varnothing \quad (\text{for } A \neq \{x\}).$$

Hence, for singletons $A = \{x\} \subseteq X$, we recover exactly $\widetilde{P}(x)$. The constraint $T + C + R + U + F \leq 5$ is maintained. Therefore, any HPNS is subsumed in n-SHPNS with $n = 1$. □

Theorem 2.21. *Every n-SuperHyperneutrosophic Set is a special case of an n-SuperHyper Pentapartitioned Neutrosophic Set by reducing two membership components to zero (e.g., $C = R = 0$).*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ maps each $A \in \mathcal{P}_n(X)$ to a subset of $[0,1]^3$ (triplets (T, I, F)). Compare with Definition 2.19, where $\widetilde{P}_n(A)$ is a subset of $[0,1]^5$ with $T + C + R + U + F \leq 5$. To recover an SHNS from n-SHPNS, we set $C = R = 0$ and identify $U = I$. Hence $(T, C, R, U, F) = (T, 0, 0, I, F)$ with $T + I + F \leq 3$. Formally, define

$$\widetilde{P}_n(A) = \Big\{ (T, 0, 0, I, F) \,:\, (T, I, F) \in \tilde{A}_n(A) \Big\}.$$

Hence, ignoring or zeroing out two membership parts yields a standard 3-part membership in $[0,1]^3$. Thus, each SHNS is included in an n-SHPNS by discarding extra membership dimensions. □

2.4 Hyper Heptapartitioned Neutrosophic Set

A Heptapartitioned Neutrosophic Set assigns seven membership [14, 44, 80–82]. This is extended using Hyperneutrosophic Sets and SuperHyperneutrosophic Sets.

Definition 2.22 (Heptapartitioned Neutrosophic Set)**.** [14, 81] A Heptapartitioned Neutrosophic Set (HNS) on a universe X is defined as:

$$HNS = \{\langle x, T(x), C(x), R(x), U(x), F(x), G(x), L(x)\rangle \mid x \in X\},$$

where $T(x), C(x), R(x), U(x), F(x), G(x), L(x) \in [0, 1]$, and

$$0 \leq T(x) + C(x) + R(x) + U(x) + F(x) + G(x) + L(x) \leq 7.$$

Definition 2.23 (Hyper Heptapartitioned Neutrosophic Set (HHNS))**.** Let X be a non-empty set, and let $\mathcal{P}([0, 1]^7)$ denote the family of all non-empty subsets of the 7-cube $[0, 1]^7$. A *Hyper Heptapartitioned Neutrosophic Set (HHNS)* $\widetilde{H}$ on X is a mapping

$$\widetilde{H} : X \longrightarrow \mathcal{P}([0, 1]^7),$$

such that for each $x \in X$,

$$\widetilde{H}(x) \subseteq \left\{ (T, C, R, U, F, G, L) \in [0, 1]^7 : T + C + R + U + F + G + L \leq 7 \right\}.$$

Hence, every point $x \in X$ is assigned a *set* of heptapartitioned membership 7-tuples (T, C, R, U, F, G, L), each lying in $[0, 1]^7$ with $T + C + R + U + F + G + L \leq 7$.

Theorem 2.24. *Every Heptapartitioned Neutrosophic Set is a special case of a Hyper Heptapartitioned Neutrosophic Set.*

Proof. A *Heptapartitioned Neutrosophic Set (HptNS)* H assigns each $x \in X$ exactly one 7-tuple $(T, C, R, U, F, G, L) \in [0, 1]^7$ with $T + C + R + U + F + G + L \leq 7$. To embed this in Definition 2.23, define

$$\widetilde{H}(x) = \Big\{\big(T(x),\ C(x),\ R(x),\ U(x),\ F(x),\ G(x),\ L(x)\big)\Big\},$$

i.e. a singleton set in $[0, 1]^7$. Since the same $T+C+R+U+F+G+L \leq 7$ constraint remains, each single-valued HptNS is embedded in HHNS as a degenerate membership set (a singleton). □

Theorem 2.25. *Every HyperNeutrosophic Set is a special case of a Hyper Heptapartitioned Neutrosophic Set by ignoring four of the membership components.*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps each $x \in X$ to a subset of $[0, 1]^3$ (triplets (T, I, F)). In Definition 2.23, each membership is a subset of $[0, 1]^7$ with $T + C + R + U + F + G + L \leq 7$. If we fix four components to zero (e.g., $C = R = G = L = 0$) and rename $U = I$, then (T, I, F) in $[0, 1]^3$ becomes $(T, 0, 0, I, F, 0, 0)$ in $[0, 1]^7$, needing $T + 0 + 0 + I + F + 0 + 0 = T + I + F \leq 3$. Formally:

$$\widetilde{H}(x) = \Big\{\big(T, 0, 0, I, F, 0, 0\big) : (T, I, F) \in \tilde{A}(x)\Big\}.$$

Hence, ignoring the extra membership components merges an HNS into an HHNS. Therefore, an HNS is a special case of HHNS by discarding (or zeroing out) the four additional partitions. □

Definition 2.26 (n-SuperHyper Heptapartitioned Neutrosophic Set (n-SHHNS))**.** Let X be a non-empty set. Define recursively:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \geq 2).$$

Similarly, let $\mathcal{P}_n\big([0, 1]^7\big)$ denote the n-th nested family of non-empty subsets of the 7-cube $[0, 1]^7$. A *n-SuperHyper Heptapartitioned Neutrosophic Set (n-SHHNS)* is a mapping

$$\widetilde{H}_n : \mathcal{P}_n(X) \rightarrow \mathcal{P}_n\big([0, 1]^7\big),$$

such that for each $A \in \mathcal{P}_n(X)$,

$$\widetilde{H}_n(A) \subseteq \Big\{(T, C, R, U, F, G, L) \in [0, 1]^7 : T + C + R + U + F + G + L \leq 7\Big\}.$$

Hence, every n-th level subset A is assigned a *set* of membership 7-tuples $(T, C, R, U, F, G, L) \in [0, 1]^7$ with $T + C + R + U + F + G + L \leq 7$.

Theorem 2.27. *Every Hyper Heptapartitioned Neutrosophic Set is a special case of an n-SuperHyper Heptapartitioned Neutrosophic Set for* $n = 1$.

Proof. A *Hyper Heptapartitioned Neutrosophic Set (HHNS)* $\widetilde{H}$ is a mapping $X \to \mathcal{P}([0,1]^7)$ with $T + C + R + U + F + G + L \leq 7$. In Definition 2.26, set $n = 1$, so

$$\widetilde{H}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \ \to \ \mathcal{P}_1\big([0,1]^7\big) = \mathcal{P}\big([0,1]^7\big).$$

Define

$$\widetilde{H}_1(\{x\}) := \widetilde{H}(x), \quad \widetilde{H}_1(A) = \varnothing \quad (\text{for } A \neq \{x\}).$$

Hence, for singletons $A = \{x\} \subseteq X$, we recover precisely the membership sets $\widetilde{H}(x)$. Thus, every HHNS is embedded in an n-SHHNS with $n = 1$. □

Theorem 2.28. *Every n-SuperHyperneutrosophic Set is a special case of an n-SuperHyper Heptapartitioned Neutrosophic Set by ignoring four membership components (e.g.* $C = R = G = L = 0$*).*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ maps each $A \in \mathcal{P}_n(X)$ to subsets of $[0,1]^3$ with (T, I, F) membership. In Definition 2.26, we assign subsets of $[0,1]^7$ with (T, C, R, U, F, G, L), each summing to at most 7. If we require $C = R = G = L = 0$ and rename $U = I$, then (T, I, F) in $[0,1]^3$ is identified with $(T, 0, 0, I, F, 0, 0)$ in $[0,1]^7$. So define

$$\widetilde{H}_n(A) = \Big\{\big(T, 0, 0, I, F, 0, 0\big) : (T, I, F) \in \tilde{A}_n(A)\Big\}.$$

Hence, ignoring four extra membership dimensions reverts us to $(T, I, F) \in [0,1]^3$. Therefore, an SHNS is included in n-SHHNS by collapsing the four additional components. □

2.5 m-Polar Hyperneutrosophic Set

An *m-Polar Hyperneutrosophic Set* extends the conventional neutrosophic framework by assigning m distinct truth, indeterminacy, and falsity triplets to each element in a given universe. This model provides a comprehensive structure for handling complex and multidimensional uncertainties [67, 75, 78, 83, 95, 98, 99, 102]. Related concepts include the bipolar neutrosophic set [1, 24, 77, 118], bipolar fuzzy set [3, 4, 66], tripolar fuzzy set [89–91], and m-polar fuzzy set [18, 63, 92, 93], which address specific dimensions of uncertainty and vagueness.

The m-Polar Neutrosophic Set is further extended using the frameworks of Hyperneutrosophic Sets and SuperHyperneutrosophic Sets, allowing for even more flexible and detailed representations of complex systems.

Definition 2.29 (m-Polar Neutrosophic Set)**.** (cf. [83, 95, 98, 99, 102]) Let X be a universe of discourse and $m \geq 1$ represent the number of poles or criteria. An m-polar neutrosophic set A is defined as:

$$A = \left\{\langle x, (T_A^{(k)}(x), I_A^{(k)}(x), F_A^{(k)}(x))\rangle \mid x \in X, k = 1, 2, \ldots, m\right\}, AbdelBasset2019CosineSM,$$

where:

- $T_A^{(k)}(x), I_A^{(k)}(x), F_A^{(k)}(x) \in [0,1]$,
- $T_A^{(k)}(x) + I_A^{(k)}(x) + F_A^{(k)}(x) \leq 3, \ \forall x \in X, \ k = 1, 2, \ldots, m.$

Definition 2.30 (m-Polar Hyperneutrosophic Set (m-HNS))**.** Let X be a non-empty set, and let $m \geq 1$. Consider the family $\mathcal{P}\big([0,1]^3\big)$ of all non-empty subsets of the unit cube $[0,1]^3$. An *m-Polar Hyperneutrosophic Set* $\widetilde{M}$ on X is defined by a mapping

$$\widetilde{M} : X \times \{1, 2, \ldots, m\} \ \to \ \mathcal{P}([0,1]^3),$$

such that for each (x, k):

$$\widetilde{M}(x, k) \ \subseteq \ \big\{(T, I, F) \in [0,1]^3 : T + I + F \leq 3\big\}.$$

Hence, each element $x \in X$ and each pole k is assigned a *set* of possible triplets (T, I, F).

Theorem 2.31. *Every m-Polar Neutrosophic Set is a special case of an m-Polar Hyperneutrosophic Set.*

Proof. An *m-Polar Neutrosophic Set (mPNS)* A assigns each (x, k) exactly one triplet $\big(T_A^{(k)}(x), I_A^{(k)}(x), F_A^{(k)}(x)\big)$ with $T + I + F \leq 3$. In Definition 2.30, we let each (x, k) map to *a set* in $[0, 1]^3$. Define

$$\widetilde{M}(x, k) = \Big\{\big(T_A^{(k)}(x), I_A^{(k)}(x), F_A^{(k)}(x)\big)\Big\},$$

a singleton. The usual constraint $T + I + F \leq 3$ holds. Hence, every mPNS is embedded in an m-Polar Hyperneutrosophic Set as a degenerate case (singleton membership for each (x, k)). □

Theorem 2.32. *Every HyperNeutrosophic Set is a special case of an m-Polar Hyperneutrosophic Set by setting* $m = 1$ *(only one pole).*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ is a mapping $X \to \mathcal{P}([0, 1]^3)$. In Definition 2.30, we have $\widetilde{M} : X \times \{1, 2, \ldots, m\} \to \mathcal{P}([0, 1]^3)$. If $m = 1$, then for each x we define

$$\widetilde{M}(x, 1) = \tilde{A}(x).$$

Hence, ignoring the multiple poles (just $k = 1$) yields exactly a HyperNeutrosophic Set. Therefore, an HNS is a sub-case of an m-Polar Hyperneutrosophic Set with $m = 1$. □

Definition 2.33 (m-Polar n-SuperHyperneutrosophic Set (m-SHNS))**.** Let X be a non-empty set, $m \geq 1$ be the number of poles, and define recursively:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \geq 2).$$

Similarly, consider $\mathcal{P}_n([0, 1]^3)$ for n-nested subsets of $[0, 1]^3$. An *m-Polar n-SuperHyperneutrosophic Set* $\widetilde{M}_n$ is given by a mapping

$$\widetilde{M}_n : \mathcal{P}_n(X) \times \{1, 2, \ldots, m\} \longrightarrow \mathcal{P}_n\big([0, 1]^3\big),$$

such that for each $(A, k) \in \mathcal{P}_n(X) \times \{1, \ldots, m\}$:

$$\widetilde{M}_n(A, k) \subseteq \big\{(T, I, F) \in [0, 1]^3 : T + I + F \leq 3\big\}.$$

Hence, each n-th level subset A and each pole k is assigned a *set* of membership triplets $(T, I, F) \in [0, 1]^3$.

Theorem 2.34. *Every m-Polar Hyperneutrosophic Set is a special case of an m-Polar n-SuperHyperneutrosophic Set for* $n = 1$.

Proof. An *m-Polar Hyperneutrosophic Set* $\widetilde{M}$ (Definition 2.30) is a mapping:

$$X \times \{1, \ldots, m\} \to \mathcal{P}([0, 1]^3).$$

In Definition 2.33, for $n = 1$ we have $\widetilde{M}_1 : \mathcal{P}_1(X) \times \{1, \ldots, m\} \to \mathcal{P}_1([0, 1]^3) = \mathcal{P}([0, 1]^3)$. We can embed $\widetilde{M}$ by defining:

$$\widetilde{M}_1(\{x\}, k) := \widetilde{M}(x, k), \quad \widetilde{M}_1(A, k) = \varnothing \quad \text{if } A \neq \{x\}.$$

Hence, for singletons $\{x\} \subset X$, we recover exactly the membership sets from $\widetilde{M}(x, k)$. The constraint $T+I+F \leq 3$ remains identical. Therefore, $\widetilde{M}$ is included in $\widetilde{M}_1$, which is an m-Polar 1-SuperHyperneutrosophic Set. □

Theorem 2.35. *Every n-SuperHyperneutrosophic Set is a special case of an m-Polar n-SuperHyperneutrosophic Set by letting* $m = 1$ *(only one pole).*

Proof. An *n-SuperHyperneutrosophic Set (SHNS)* $\tilde{A}_n$ is a mapping $\mathcal{P}_n(X) \to \mathcal{P}_n([0, 1]^3)$. In Definition 2.33, we have $\widetilde{M}_n : \mathcal{P}_n(X) \times \{1, \ldots, m\} \to \mathcal{P}_n([0, 1]^3)$. If $m = 1$, we define

$$\widetilde{M}_n(A, 1) = \tilde{A}_n(A), \quad \widetilde{M}_n(A, k) = \varnothing \quad \text{for } k \neq 1.$$

Hence, ignoring multiple poles (just $k = 1$) we recover the usual n-SuperHyperneutrosophic mapping from $\mathcal{P}_n(X)$ to $\mathcal{P}_n([0, 1]^3)$. Therefore, an SHNS is embedded in m-SHNS with $m = 1$. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Mohamed Abdel-Basset, Mai Mohamed, Mohamed Elhoseny, Le Hoang Son, Francisco Chiclana, and Abdel Nasser H. Zaied. Cosine similarity measures of bipolar neutrosophic set for diagnosis of bipolar disorder diseases. *Artificial intelligence in medicine*, 101:101735, 2019.

[2] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[3] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.

[4] Muhammad Akram, Shumaiza, and José Carlos Rodríguez Alcantud. Multi-criteria decision making methods with bipolar fuzzy sets. *Forum for Interdisciplinary Mathematics*, 2023.

[5] Manal Al-Labadi, Shuker Khalil, VR Radhika, K Mohana, et al. Pentapartitioned neutrosophic vague soft sets and its applications. *International Journal of Neutrosophic Science*, (2):64–4, 2025.

[6] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[7] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[8] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[9] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[10] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[11] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[12] Vivek Badhe, RS Thakur, and GS Thakur. Vague set theory for profit pattern and decision making in uncertain data. *International journal of advanced computer science and applications*, 6(6):58–64, 2015.

[13] Pranab Biswas, Surapati Pramanik, and Bibhas Chandra Giri. Single valued bipolar pentapartitioned neutrosophic set and its application in madm strategy. 2022.

[14] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[15] Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.

[16] Rajashi Chatterjee, Pinaki Majumdar, and Syamal Kumar Samanta. A multi-criteria group decision making algorithm with quadripartitioned neutrosophic weighted aggregation operators using quadripartitioned neutrosophic numbers in ipqsvnss environment. *Soft Computing*, 24(12):8857–8880, 2020.

[17] Tanmoy Chatterjee and Surapati Pramanik. Triangular fuzzy quadripartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 75:15–28, 2025.

[18] Juanjuan Chen, Shenggang Li, Shengquan Ma, and Xueping Wang. m-polar fuzzy sets: an extension of bipolar fuzzy sets. *The scientific world journal*, 2014(1):416530, 2014.

[19] Wen-Hua Cui and Jun Ye. Logarithmic similarity measure of dynamic neutrosophic cubic sets and its application in medical diagnosis. *Computers in Industry*, 111:198–206, 2019.

[20] Rakhal Das and Suman Das. Pentapartitioned neutrosophic subtraction algebra. *Neutrosophic Sets and Systems*, 68:89–98, 2024.

[21] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[22] Suman Das, Prasanna Poojary, Vadiraja Bhatta GR, and Sunny Kumar Sharma. Single-valued pentapartitioned neutrosophic bi-topological spaces. *International Journal of Neutrosophic Science (IJNS)*, 23(4), 2024.

[23] Somen Debnath. Fuzzy quadripartitioned neutrosophic soft matrix theory and its decision-making approach. *Journal of Computational and Cognitive Engineering*, 1(2):88–93, 2022.

[24] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[25] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[26] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[27] Takaaki Fujita. Natural n-superhyper plithogenic language.

[28] Takaaki Fujita. Review of plithogenic directed, mixed, bidirected, and pangene offgraph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 120.

[29] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[30] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[31] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[32] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[33] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[34] Takaaki Fujita. Short communication of big plithogenic science and deep plithogenic science. 2024. preprint (researchgate).

[35] Takaaki Fujita. Short note of bunch graph in fuzzy, neutrosophic, and plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2024.

[36] Takaaki Fujita. Some graph classes for turiyam neutrosophic directed graphs and pentapartitioned neutrosophic directed graphs. 2024.

[37] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[38] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[39] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[40] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). July 2024.

[41] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[42] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[43] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[44] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[45] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[46] Takaaki Fujita. Short note of even-hole-graph for uncertain graph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 351, 2025.

[47] Takaaki Fujita. Short note of extended hyperplithogenic sets, 2025. Preprint.

[48] Takaaki Fujita. A short note on the basic graph construction algorithm for plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 274, 2025.

[49] Takaaki Fujita. Short survey on the hierarchical uncertainty of fuzzy, neutrosophic, and plithogenic sets, 2025. Preprint.

[50] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[51] Takaaki Fujita. Some types of hyperneutrosophic set (2): Complex, single-valued triangular, fermatean, and linguistic sets. *Preprint*, 2025.

[52] Takaaki Fujita and Florentin Smarandache. A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 308.

[53] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[54] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[55] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[56] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[57] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[58] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[59] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[60] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[61] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[62] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[63] Harish Garg, Muhammad Riaz, Muhammad Abdullah Khokhar, and Maryam Saba. Correlation measures for cubic m-polar fuzzy sets with applications. *Mathematical Problems in Engineering*, 2021.

[64] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[65] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[66] Muhammad Gulistan, Naveed Yaqoob, Ahmed Elmoasry, and Jawdat Alebraheem. Complex bipolar fuzzy sets: An application in a transport's company. *J. Intell. Fuzzy Syst.*, 40:3981–3997, 2021.

[67] Masooma Raza Hashmi, Muhammad Riaz, and Florentin Smarandache. m-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis. *International Journal of Fuzzy Systems*, 22:273–292, 2020.

[68] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[69] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[70] Satham S Hussain, Muhammad Aslam, Hossein Rahmonlou, and N Durga. Applying interval quadripartitioned single-valued neutrosophic sets to graphs and climatic analysis. In *Data-Driven Modelling with Fuzzy Sets*, pages 100–143. CRC Press, 2025.

[71] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[72] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[73] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[74] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[75] M. A. Malik. m-polar neutrosophic graphs. 2021.

[76] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[77] Mai Mohamed and Asmaa Elsayed. A novel multi-criteria decision making approach based on bipolar neutrosophic set for evaluating financial markets in egypt. *Multicriteria Algorithms with Applications*, 2024.

[78] Kartick Mohanta, Arindam Dey, Anita Pal, Hoang Viet Long, and Le Hoang Son. A study of m-polar neutrosophic graph with applications. *Journal of Intelligent & Fuzzy Systems*, 38(4):4809–4828, 2020.

[79] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[80] T Mythili, V Jeyanthi, and KR Senthilkumar. Enhanced decision-making academic libraries with topsis methods using blockchain technology by heptapartitioned neutrosophic number. In *Leveraging Blockchain for Future-Ready Libraries*, pages 123–140. IGI Global Scientific Publishing, 2025.

[81] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[82] M Myvizhi, Mohamed Elkholy, Ahmed Abdelhafeez, Hussam Elbehiery, et al. Enhanced madm strategy with heptapartitioned neutrosophic distance metrics. *Neutrosophic Sets and Systems*, 78:74–96, 2025.

[83] Khalid Naeem and Bijan Divvaz. Information measures for madm under m-polar neutrosophic environment. *Granular Computing*, 8(3):597–616, 2023.

[84] Junda Qiu, Linjia Jiang, Honghui Fan, Peng Li, and Congzhe You. Dynamic nonlinear simplified neutrosophic sets for multiple-attribute group decision making. *Heliyon*, 10(5), 2024.

[85] Radha R. and Stanis Arul Mary. Pentapartitioned neutrosophic pythagorean set. *International Research Journal on Advanced Science Hub*, 2021.

[86] R. Radha, A. Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science*, 2021.

[87] Ramakrishnan Radha, A. Stanis, and Arul Mary. Pentapartitioned neutrosophic pythagorean soft set. 2021.

[88] M Ramya, Sandesh Murali, and R.Radha. Bipolar quadripartitioned neutrosophic soft set. 2022.

[89] M Murali Krishna Rao. Tripolar fuzzy interior ideals and tripolar fuzzy soft interior ideals over semigroups. *Annals of Fuzzy Mathematics and Informatics*, 20(3):243–256, 2020.

[90] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[91] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, (422019):731743, 2019.

[92] Muhammad Riaz, Muhammad Tahir Hamid, Deeba Afzal, Dragan Pamucar, and Yuming Chu. Multi-criteria decision making in robotic agri-farming with q-rung orthopair m-polar fuzzy sets. *PLoS ONE*, 16, 2021.

[93] Muhammad Riaz and Masooma Raza Hashmi. Soft rough pythagorean m-polar fuzzy sets and pythagorean m-polar fuzzy soft rough sets with application to decision-making. *Computational and Applied Mathematics*, 39, 2019.

[94] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[95] Muhammad Saeed, Muhammad Saqlain, and Asad Mehmood. *Application of similarity measure on m-polar interval-valued neutrosophic set in decision making in sports*, volume 38. Infinite Study, 2020.

[96] AA Salama, KF Alhasan, HA Elagamy, and Florentin Smarandache. Neutrosophic dynamic set. *Neutrosophic Knowledge*, 3, 2021.

[97] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[98] Muhammad Saqlain and Xiao Long Xin. *Interval valued, m-polar and m-polar interval valued neutrosophic hypersoft sets*. Infinite Study, 2020.

[99] S Satham Hussain, Durga Nagarajan, Hossein Rashmanlou, and Farshid Mofidnakhaei. Novel supply chain decision making model under m-polar quadripartitioned neutrosophic environment. *Journal of Applied Mathematics and Computing*, pages 1–26, 2024.

[100] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[101] Bimal Shil, Rakhal Das, Carlos Granados, Suman Das, and Binod Chandra Tripathy. Single-valued quadripartitioned neutrosophic d-ideal of d-algebra. *Neutrosophic Sets and Systems*, 67:105–114, 2024.

[102] M Sivakumar, Abdul Rajak Rabıyathul Basarıya, M Senthil, G Raja, R Rajavarman, et al. Transforming arabic text analysis: Integrating applied linguistics with m-polar neutrosophic set mood change and depression on social media. *International Journal of Neutrosophic Science*, (2):313–13, 2025.

[103] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[104] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[105] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[106] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[107] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[108] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[109] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[110] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[111] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[112] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[113] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[114] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[115] Nguyen Tho Thong, Luong Thi Hong Lan, Shuo-Yan Chou, Le Hoang Son, Do Duc Dong, and Tran Thi Ngan. An extended topsis method with unknown weight information in dynamic neutrosophic environment. *Mathematics*, 8(3):401, 2020.

[116] Nguyen Tho Thong, Luu Quoc Dat, Nguyen Dinh Hoa, Mumtaz Ali, Florentin Smarandache, et al. Dynamic interval valued neutrosophic set: Modeling decision making in dynamic environments. *Computers in Industry*, 108:45–52, 2019.

[117] Nguyen Tho Thong, Florentin Smarandache, Nguyen Dinh Hoa, Le Hoang Son, Luong Thi Hong Lan, Cu Nguyen Giap, Dao The Son, and Hoang Viet Long. A novel dynamic multi-criteria decision making method based on generalized dynamic interval-valued neutrosophic set. *Symmetry*, 12(4):618, 2020.

[118] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[119] Jun Ye. Correlation coefficient between dynamic single valued neutrosophic multisets and its multiple attribute decision-making method. *Information*, 8(2):41, 2017.

[120] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[121] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[122] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[123] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[124] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

Chapter 6

Some Types of HyperNeutrosophic Set (4): Cubic, Trapozoidal, q-Rung Orthopair, Overset, Underset, and Offset

Takaaki Fujita [1] * and Florentin Smarandache[2],
[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp
[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

This paper builds upon the foundational work presented in [38–40]. The Neutrosophic Set provides a comprehensive mathematical framework for managing uncertainty, defined by three membership functions: truth, indeterminacy, and falsity. Recent advancements have introduced extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set, which are specifically designed to address increasingly complex and multidimensional problems. The formal definitions of these sets are available in [30].

In this paper, we extend the Neutrosophic Cubic Set, Trapezoidal Neutrosophic Set, q-Rung Orthopair Neutrosophic Set, Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset using the frameworks of the Hyperneutrosophic Set and the SuperHyperneutrosophic Set. Furthermore, we briefly examine their properties and potential applications.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. The analysis utilizes classical set-theoretic operations and extends them into advanced frameworks. For readers seeking a deeper understanding of foundational set theory, resources such as [16, 52, 55, 60] are recommended. Additionally, the referenced literature offers a comprehensive exploration of the principles and applications of Neutrosophic Sets.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To address uncertainty, vagueness, and imprecision in decision-making processes, numerous set-theoretic frameworks have been developed. These frameworks include Fuzzy Sets, which were introduced in foundational works such as those by Zadeh [105–109]. Another prominent framework is Intuitionistic Fuzzy Sets, extensively studied by Atanassov and others [5–10]. Vague Sets, introduced and developed by researchers, also contribute significantly to this domain [1, 11, 49, 63, 74].

More recently, Plithogenic Sets, as proposed and expanded by Smarandache, have gained attention for their ability to model complex scenarios involving contradictions and multi-dimensional uncertainty [18, 24, 26–28, 36, 37, 46, 85, 87, 88]. Soft Sets, as introduced by Molodtsov and further studied by other scholars, provide a flexible mathematical tool for handling uncertainty [50, 64, 67].

Additionally, Hypersoft Sets, an extension of Soft Sets, have been explored in various applications by Smarandache [20, 31, 45, 86]. Neutrosophic Sets, first introduced by Smarandache, offer a powerful means of capturing indeterminacy, allowing for more nuanced decision-making models [21, 22, 25, 35, 41–44, 47, 48, 79, 80, 94]. Neutrosophic Sets generalize Fuzzy Sets by introducing an additional component: indeterminacy, alongside truth and falsity [77–80]. This enhancement allows for a richer and more precise representation of uncertainty and ambiguity.

To address even more complex scenarios, the HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets have been developed. These advanced models are particularly suited for high-dimensional and intricate problem spaces [19, 30].

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [26, 73] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [17, 26, 32, 76, 91])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Neutrosophic Set)**.** [79, 80] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 1.5 (HyperNeutrosophic Set)**.** (cf. [19, 30, 33, 34, 84]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0, 1]^3),$$

where $\mathcal{P}([0, 1]^3)$ is the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Definition 1.6 (n-SuperHyperNeutrosophic Set)**.** (cf. [19, 30, 33, 34, 84]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0, 1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0, 1]^3)$ is defined similarly for the unit cube $[0, 1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Neutrosophic Cubic Set

A Neutrosophic Cubic Set (NCS) combines Interval Neutrosophic Sets and Neutrosophic Sets, representing uncertainty through interval and point-based truth, indeterminacy, and falsity values [12,14,51,56,70,96,110].

Definition 2.1 (Neutrosophic Cubic Set (NCS))**.** [3,56] Let X be a non-empty set. A *Neutrosophic Cubic Set (NCS) A* in X is a pair $A = (A_{INS}, A_{NS})$, where:

- $A_{INS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$ is an *Interval Neutrosophic Set (INS)* in X. For each $x \in X$, $T_A(x) = [T_A^-, T_A^+]$, $I_A(x) = [I_A^-, I_A^+]$, $F_A(x) = [F_A^-, F_A^+]$, where $T_A, I_A, F_A \subseteq [0,1]$.
- $A_{NS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$ is a *Neutrosophic Set (NS)* in X. Here, $T_A, I_A, F_A : X \to [0,1]$, satisfying $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$.

The pair $A = (A_{INS}, A_{NS})$ generalizes the notions of Interval Neutrosophic Sets and Neutrosophic Sets, allowing for a hybrid representation of uncertainty.

Remark 2.2 (Neutrosophic Cubic membership domain)**.** For convenience, define the *Neutrosophic Cubic membership domain* $C \subseteq [0,1]^9$ by:

$$C = \left\{ \left(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F\right) \in [0,1]^9 : \begin{array}{l} 0 \leq T^- \leq T \leq T^+ \leq 1, \\ 0 \leq I^- \leq I \leq I^+ \leq 1, \\ 0 \leq F^- \leq F \leq F^+ \leq 1, \\ (T^- + I^- + F^-) \leq 3, \quad (T^+ + I^+ + F^+) \leq 3, \quad (T + I + F) \leq 3 \end{array} \right\}.$$

Each 9-tuple in C represents both *interval* membership (the triple intervals $[T^-, T^+]$, $[I^-, I^+]$, $[F^-, F^+]$) and *point* membership (T, I, F), subject to usual neutrosophic constraints.

Definition 2.3 (HyperNeutrosophic Cubic Set (HNCS))**.** Let X be a non-empty set, and let $\mathcal{P}(C)$ be the family of all non-empty subsets of the domain $C \subseteq [0,1]^9$ (as defined above). A *HyperNeutrosophic Cubic Set (HNCS) $\widetilde{N}$* on X is a mapping

$$\widetilde{N} : X \longrightarrow \mathcal{P}(C),$$

where for each $x \in X$, $\widetilde{N}(x)$ is a *set* of 9-tuples

$$\left(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F\right) \in C$$

satisfying the constraints for Neutrosophic Cubic membership (i.e. $T^- \leq T \leq T^+, T^- + I^- + F^- \leq 3$, etc.).

Hence, each point x may have *multiple* possible cubic memberships, capturing a range (hyper-set) of intervals plus point-based membership data.

Theorem 2.4. *Every Neutrosophic Cubic Set is a special case of a HyperNeutrosophic Cubic Set.*

Proof. A *Neutrosophic Cubic Set (NCS) A* over X assigns each $x \in X$ a single pair $(A_{INS}(x), A_{NS}(x))$ of interval membership plus point membership. Concretely, it can be described by a single 9-tuple

$$\left(T_A^-(x), T_A^+(x),\ I_A^-(x), I_A^+(x),\ F_A^-(x), F_A^+(x),\ T_A(x), I_A(x), F_A(x)\right) \in C.$$

In Definition 2.3, an HNCS is a mapping $\widetilde{N} : X \to \mathcal{P}(C)$. We embed A by letting

$$\widetilde{N}(x) = \left\{\left(T_A^-(x), T_A^+(x),\ I_A^-(x), I_A^+(x),\ F_A^-(x), F_A^+(x),\ T_A(x), I_A(x), F_A(x)\right)\right\},$$

a *singleton* in C. Since all constraints on intervals and points match those in the domain C, A is reproduced exactly. Thus, every NCS is a degenerate (single membership) version of an HNCS. □

Theorem 2.5. *Every HyperNeutrosophic Set is a special case of a HyperNeutrosophic Cubic Set by collapsing the interval portion to a single point.*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps each $x \in X$ to a subset of $[0,1]^3$ with $T + I + F \leq 3$. In Definition 2.3, we use $C \subseteq [0,1]^9$. If we force $T^- = T = T^+$, $I^- = I = I^+$, $F^- = F = F^+$, then the 9-tuple

$$(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F)$$

collapses to $(T, T, T, I, I, I, F, F, F)$ with $T + I + F \leq 3$. This effectively recovers a 3D membership (T,I,F). Formally, define

$$\widetilde{N}(x) = \Big\{(T, T, T, I, I, I, F, F, F) \ \mid \ (T, I, F) \in \tilde{A}(x)\Big\}.$$

Hence, ignoring the intervals (merging them with the single values) yields a standard hyperneutrosophic membership. Therefore, an HNS is embedded in an HNCS by collapsing intervals to single points. □

Definition 2.6 (n-SuperHyperNeutrosophic Cubic Set (n-SHNCS))**.** Let X be a non-empty set. Define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \geq 2).$$

Likewise, define $\mathcal{P}_n(C)$ as the n-th nested power set of the cubic domain $C \subseteq [0,1]^9$ from above. An *n-SuperHyperNeutrosophic Cubic Set (n-SHNCS)* is a mapping

$$\widetilde{N}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n(C),$$

such that for each $A \in \mathcal{P}_n(X)$, $\widetilde{N}_n(A) \subseteq C$. Concretely, each n-th level subset A in X is assigned a *set* of 9-tuples

$$\big(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F\big) \in C,$$

all obeying the neutrosophic cubic constraints (interval plus point membership).

Theorem 2.7. *Every HyperNeutrosophic Cubic Set is a special case of an n-SuperHyperNeutrosophic Cubic Set for* $n = 1$.

Proof. A *HyperNeutrosophic Cubic Set (HNCS)* $\widetilde{N}$ is a mapping $X \to \mathcal{P}(C)$. In Definition 2.6, if we set $n = 1$, we get

$$\widetilde{N}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \longrightarrow \mathcal{P}_1(C) = \mathcal{P}(C).$$

We define

$$\widetilde{N}_1(\{x\}) = \widetilde{N}(x), \quad \widetilde{N}_1(A) = \varnothing \quad \text{for } A \neq \{x\}.$$

Hence, for singletons $A = \{x\}$, $\widetilde{N}_1(\{x\})$ recovers exactly the membership set $\widetilde{N}(x)$ in C. The same constraints remain. Therefore, every HNCS is included in an n-SuperHyperNeutrosophic Cubic Set with $n = 1$. □

Theorem 2.8. *Every n-SuperHyperNeutrosophic Set is a special case of an n-SuperHyperNeutrosophic Cubic Set by collapsing the interval membership to single points.*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ maps each $A \in \mathcal{P}_n(X)$ to subsets of $[0,1]^3$, each triple (T, I, F) satisfying $T + I + F \leq 3$. In Definition 2.6, $\widetilde{N}_n(A)$ is a subset of the domain $C \subseteq [0,1]^9$. To recover an SHNS from n-SHNCS, we identify $(T^-, T^+, I^-, I^+, F^-, F^+, T, I, F)$ with $(T, T, I, I, F, F, T, I, F)$ in which $T^- = T^+ = T$, $I^- = I^+ = I$, and $F^- = F^+ = F$. Then $T + I + F \leq 3$ is the standard constraint. Formally:

$$\widetilde{N}_n(A) = \Big\{(T, T, I, I, F, F, T, I, F) \ \mid \ (T, I, F) \in \tilde{A}_n(A)\Big\}.$$

Hence, ignoring intervals or collapsing them to single points recovers a 3D membership in $[0,1]^3$. Thus, an SHNS is embedded in n-SHNCS by dropping the interval portion. □

2.2 Trapezoidal Neutrosophic Set

A Trapezoidal Neutrosophic Set (TNS) utilizes trapezoidal fuzzy numbers to represent truth, indeterminacy, and falsity memberships, enabling advanced modeling of uncertainty [4,13,57,59,101,102]. A closely related concept is the Trapezoidal Fuzzy Set [61,66,100,103,104].

Definition 2.9 (Trapezoidal Neutrosophic Set)**.** [101] A *Trapezoidal Neutrosophic Set (TNS) A* in a universe of discourse X is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\},$$

where:

$$T_A(x) = (t_1, t_2, t_3, t_4), \quad I_A(x) = (i_1, i_2, i_3, i_4), \quad F_A(x) = (f_1, f_2, f_3, f_4),$$

are *trapezoidal fuzzy numbers* that represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. These functions satisfy the following conditions:

$$t_1 \le t_2 \le t_3 \le t_4, \quad i_1 \le i_2 \le i_3 \le i_4, \quad f_1 \le f_2 \le f_3 \le f_4,$$

and

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \quad \forall x \in X.$$

Each trapezoidal membership function is defined piecewise:

$$T_A(x) = \begin{cases} \frac{x-t_1}{t_2-t_1}, & t_1 \le x \le t_2, \\ 1, & t_2 \le x \le t_3, \\ \frac{t_4-x}{t_4-t_3}, & t_3 \le x \le t_4, \\ 0, & \text{otherwise.} \end{cases}$$

The indeterminacy-membership $I_A(x)$ and falsity-membership $F_A(x)$ follow similar definitions with their respective parameters.

Remark 2.10 (Trapezoidal Neutrosophic domain)**.** To handle trapezoids and the neutrosophic constraint, define the *Trapezoidal Neutrosophic domain*:

$$\mathcal{T} \subseteq ([0,1]^4)^3$$

where each triple $\big((t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4)\big)$ must satisfy

$$t_1 \le t_2 \le t_3 \le t_4, \quad i_1 \le i_2 \le i_3 \le i_4, \quad f_1 \le f_2 \le f_3 \le f_4,$$

and possibly a constraint like $T_A(x) + I_A(x) + F_A(x) \le 3$ in an integrated sense (though exact interpretation can vary). For simplicity, we can embed the trapezoid-based membership directly, assuming each trapezoid is in $[0,1]^4$ with ascending coordinates.

Definition 2.11 (Trapezoidal HyperNeutrosophic Set (THNS))**.** Let X be a non-empty set, and let $\mathcal{P}(\mathcal{T})$ be the family of all non-empty subsets of the trapezoidal domain $\mathcal{T} \subseteq ([0,1]^4)^3$. A *Trapezoidal HyperNeutrosophic Set (THNS)* $\widetilde{T}$ on X is a mapping

$$\widetilde{T} : X \longrightarrow \mathcal{P}(\mathcal{T}),$$

such that for each $x \in X$, $\widetilde{T}(x)$ is a *set* of trapezoidal triplets

$$\big((t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4)\big) \in \mathcal{T},$$

capturing multiple possible trapezoidal membership functions for truth, indeterminacy, and falsity. Each triple of trapezoids is typically constrained by $0 \le t_1 \le t_2 \le t_3 \le t_4 \le 1$, etc., and respects a neutrosophic boundary (e.g. up to ≤ 3 in some integrated sense).

Theorem 2.12. *Every Trapezoidal Neutrosophic Set is a special case of a Trapezoidal HyperNeutrosophic Set.*

Proof. A *Trapezoidal Neutrosophic Set (TNS) A* assigns each $x \in X$ exactly one triple of trapezoids $(T_A(x), I_A(x), F_A(x)) \in ([0,1]^4)^3$. In Definition 2.11, we define $\widetilde{T}(x) \subseteq \mathcal{T}$. We embed A by letting

$$\widetilde{T}(x) = \Big\{\big(T_A(x), I_A(x), F_A(x)\big)\Big\},$$

a singleton set. This precisely recovers the TNS membership. Hence, every TNS is embedded in THNS as a degenerate (single membership) case. □

Theorem 2.13. *Every HyperNeutrosophic Set is a special case of a Trapezoidal HyperNeutrosophic Set by collapsing trapezoids to single numeric values.*

Proof. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ maps $x \in X$ to subsets of $[0,1]^3$, each triple (T, I, F) with $T + I + F \leq 3$. In Definition 2.11, each membership is in $\mathcal{T} \subseteq ([0,1]^4)^3$. If we set $t_1 = t_2 = t_3 = t_4 = T$, $i_1 = i_2 = i_3 = i_4 = I$, $f_1 = f_2 = f_3 = f_4 = F$, each trapezoid degenerates to a single point. Formally:

$$\widetilde{T}(x) = \Big\{\big((T,T,T,T), (I,I,I,I), (F,F,F,F)\big) \;\big|\; (T,I,F) \in \tilde{A}(x)\Big\}.$$

Thus, ignoring the trapezoidal range merges the set into numeric values. Hence, an HNS emerges as a special (collapsed trapezoid) case of THNS. □

Definition 2.14 (Trapezoidal n-SuperHyperNeutrosophic Set (T-n-SHNS))**.** Let X be a non-empty set. Define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big) \quad (k \geq 2).$$

Similarly, let $\mathcal{P}_n(\mathcal{T})$ denote the n-th nested power set of the trapezoidal domain $\mathcal{T} \subseteq ([0,1]^4)^3$. A *Trapezoidal n-SuperHyperNeutrosophic Set (T-n-SHNS)* is a mapping

$$\widetilde{T}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n(\mathcal{T}),$$

meaning for each $A \in \mathcal{P}_n(X)$, $\widetilde{T}_n(A) \subseteq \mathcal{T}$. Concretely, each n-th level subset A is assigned a *set* of trapezoidal membership triples

$$\big(T_A(x), I_A(x), F_A(x)\big) \in \big([0,1]^4\big)^3,$$

satisfying the trapezoidal ordering constraints and a neutrosophic boundary (e.g. up to ≤ 3 in some integrated sense).

Theorem 2.15. *Every Trapezoidal HyperNeutrosophic Set is a special case of a Trapezoidal n-SuperHyperNeutrosophic Set (T-n-SHNS) for $n = 1$.*

Proof. A *Trapezoidal HyperNeutrosophic Set (THNS)* $\widetilde{T}$ (Definition 2.11) is a mapping $X \to \mathcal{P}(\mathcal{T})$. In Definition 2.14, for $n = 1$ we have

$$\widetilde{T}_1 : \mathcal{P}_1(X) = \mathcal{P}(X) \to \mathcal{P}_1(\mathcal{T}) = \mathcal{P}(\mathcal{T}).$$

We embed $\widetilde{T}$ by defining:

$$\widetilde{T}_1(\{x\}) := \widetilde{T}(x), \quad \widetilde{T}_1(A) = \varnothing \quad (\text{for } A \neq \{x\}).$$

Hence, each singleton $\{x\} \subseteq X$ recovers exactly $\widetilde{T}(x)$. Thus, $\widetilde{T}_1$ is a T-1-SHNS that coincides with the THNS $\widetilde{T}$. □

Theorem 2.16. *Every n-SuperHyperNeutrosophic Set is a special case of a Trapezoidal n-SuperHyperNeutrosophic Set by collapsing trapezoids to single points.*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ maps $\mathcal{P}_n(X)$ to subsets of $[0,1]^3$. In Definition 2.14, T-n-SHNS uses $\mathcal{T} \subseteq (([0,1]^4)^3)$. If we make each trapezoid degenerate, e.g. $t_1 = t_2 = t_3 = t_4 = T$, etc., we effectively recover single numeric values (T, I, F). Formally:

$$\widetilde{T}_n(A) = \Big\{\big((T,T,T,T),\ (I,I,I,I),\ (F,F,F,F)\big) \;\big|\; (T,I,F) \in \tilde{A}_n(A)\Big\}.$$

Hence, ignoring the trapezoidal range merges the membership into single numeric triplets. Thus, any n-SuperHyperNeutrosophic Set is included in T-n-SHNS by collapsing the trapezoids to single points. □

2.3 q-Rung Orthopair Neutrosophic Set

A q-Rung Orthopair Neutrosophic Set (q-RONS) generalizes orthopair sets, constraining q-th powers of truth, indeterminacy, and falsity to sum ≤ 2 [75,97,98]. Related concepts include the q-Rung Orthopair Fuzzy Set, among others [2,15,29,53,54,58,62,69,71,72,93,99].

Definition 2.17 (q-Rung Orthopair Neutrosophic Set)**.** [75,98] Let U be a universal set. A *q-Rung Orthopair Neutrosophic Set (q-RONS)* is defined as:

$$A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in U\},$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ are the truth-membership, indeterminacy-membership, and falsity-membership degrees, respectively. These satisfy:

1. $T_A(x), I_A(x), F_A(x) \in [0,1]$,
2. $$[T_A(x)]^q + [I_A(x)]^q + [F_A(x)]^q \leq 2, \quad q > 0.$$

Definition 2.18 (q-Rung Orthopair HyperNeutrosophic Set (q-RHNS))**.** Let U be a non-empty set, and let $q > 0$. A *q-Rung Orthopair HyperNeutrosophic Set (q-RHNS)* on U is a mapping

$$\widetilde{Q} : U \longrightarrow \mathcal{P}\big([0,1]^3\big),$$

where for each $x \in U$, $\widetilde{Q}(x) \subseteq [0,1]^3$ is a *set* of triplets (T, I, F), each triplet satisfying

$$T^q + I^q + F^q \;\leq\; 2, \quad (T, I, F) \in [0,1]^3.$$

Theorem 2.19. *Every q-Rung Orthopair Neutrosophic Set is a special case of a q-Rung Orthopair HyperNeutrosophic Set.*

Proof. A *q-Rung Orthopair Neutrosophic Set (q-RONS) A* on U assigns each $x \in U$ exactly one triplet $(T_A(x), I_A(x), F_A(x)) \in [0,1]^3$ with $(T_A(x))^q + (I_A(x))^q + (F_A(x))^q \leq 2$. In Definition 2.18, we let each x map to a *set* of triplets. So define:

$$\widetilde{Q}(x) = \Big\{\big(T_A(x), I_A(x), F_A(x)\big)\Big\},$$

a singleton set. The same q-rung condition persists. Hence, each q-RONS is naturally embedded in the q-RHNS framework as a degenerate (singleton) membership set. □

Theorem 2.20. *Every HyperNeutrosophic Set can be viewed as a special case of a q-Rung Orthopair HyperNeutrosophic Set by setting $q = 1$ or adjusting membership sums.*

Proof. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* maps U to subsets of $[0,1]^3$, each triplet (T, I, F) typically satisfying $T + I + F \leq 3$ or a scaled version. In Definition 2.18, we have (T, I, F) with $T^q + I^q + F^q \leq 2$. If we set $q = 1$ and rescale the boundary appropriately (like $T + I + F \leq 2$ or a linear transformation to align with ≤ 3), we can embed an HNS. Formally:

$$\widetilde{Q}(x) = \tilde{A}(x) \quad \text{with the understanding that for each } (T, I, F) \in \tilde{A}(x),\ T + I + F \leq 2,$$

or we rescale so that $T^q + I^q + F^q \leq 2$ is equivalent to $T + I + F \leq 3$ after a linear or parametric transformation. Thus, ignoring the q-rung power or setting $q = 1$ collapses q-RHNS to an HNS. □

Definition 2.21 (q-Rung Orthopair n-SuperHyperNeutrosophic Set (q-RHNS$_n$))**.** Let U be a non-empty set, $q > 0$. Define recursively:

$$\mathcal{P}_1(U) = \mathcal{P}(U), \quad \mathcal{P}_k(U) = \mathcal{P}\big(\mathcal{P}_{k-1}(U)\big) \quad (k \geq 2).$$

Similarly, consider $\mathcal{P}_n([0,1]^3)$ for the n-nested subsets of the unit cube $[0,1]^3$. A *q-Rung Orthopair n-SuperHyperNeutrosophic Set (q-RHNS$_n$)* is a mapping

$$\widetilde{Q}_n : \mathcal{P}_n(U) \;\rightarrow\; \mathcal{P}_n([0,1]^3),$$

such that for each $A \in \mathcal{P}_n(U)$, $\widetilde{Q}_n(A) \subseteq [0,1]^3$ is a *set* of triplets (T, I, F) satisfying

$$T^q + I^q + F^q \;\leq\; 2.$$

Hence, each n-th level subset A is assigned a set of q-rung orthopair membership triplets in $[0,1]^3$.

Theorem 2.22. *Every q-Rung Orthopair HyperNeutrosophic Set is a special case of a q-Rung Orthopair n-SuperHyperNeutrosophic Set for* $n = 1$.

Proof. A *q-Rung Orthopair HyperNeutrosophic Set (q-RHNS)* $\widetilde{Q}$ is a mapping $U \to \mathcal{P}([0,1]^3)$, each triplet satisfying (T, I, F) with $T^q + I^q + F^q \leq 2$. In Definition 2.21, for $n = 1$ we have:

$$\widetilde{Q}_1 : \mathcal{P}_1(U) = \mathcal{P}(U) \ \to \ \mathcal{P}_1([0,1]^3) = \mathcal{P}([0,1]^3).$$

We define:

$$\widetilde{Q}_1(\{x\}) := \widetilde{Q}(x), \quad \widetilde{Q}_1(A) = \varnothing \quad (\text{for } A \neq \{x\}).$$

Hence, for singletons $\{x\} \subset U$, we recover exactly the membership sets from $\widetilde{Q}(x)$. The q-rung condition remains. Thus, $\widetilde{Q}$ is embedded in $\widetilde{Q}_1$ as a special case. □

Theorem 2.23. *Every n-SuperHyperNeutrosophic Set can be viewed as a special case of a q-Rung Orthopair n-SuperHyperNeutrosophic Set by letting* $q = 1$ *or ignoring the q-rung power.*

Proof. An *n-SuperHyperNeutrosophic Set (SHNS)* $\tilde{A}_n$ assigns each $A \in \mathcal{P}_n(U)$ a subset of $[0,1]^3$, each (T, I, F) satisfying $T + I + F \leq 3$ or a similar constraint. In Definition 2.21, a q-RHNS$_n$ uses the condition $T^q + I^q + F^q \leq 2$. If we set $q = 1$ and adjust the boundary from 2 to 3 by a simple scaling (or interpret sum ≤ 2 as a scaled version of ≤ 3), we recover the classical n-SHNS. Formally, define

$$\widetilde{Q}_n(A) = \tilde{A}_n(A) \quad \text{with the sum constraint replaced or scaled so } (T, I, F) \text{ meet } T^q + I^q + F^q \leq 2 \text{ for } q = 1.$$

Hence, ignoring or setting $q = 1$ collapses the q-rung approach to the usual sum-based approach. Therefore, each n-SHNS can be embedded in a q-RHNS$_n$ by suitably setting $q = 1$ and matching bounds. □

2.4 Neutrosophic Overset, Underset, and Offset

Neutrosophic Overset, Underset, and Offset extend traditional neutrosophic sets. Overset includes external elements, Underset excludes specific elements, and Offset captures deviations, enhancing uncertainty and flexibility modeling [23, 65, 68, 78, 89, 90, 92, 95].

Definition 2.24 (Neutrosophic Overset)**.** [81–83] Let U be a universe of discourse, and let $T(x), I(x), F(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively. For a neutrosophic overset A, these functions satisfy:

$$T(x), I(x), F(x) : U \to [0, \Omega], \quad \Omega > 1.$$

A Neutrosophic Overset is given by:

$$A = \{(x, T(x), I(x), F(x)) \mid x \in U, \exists x \in U \text{ such that } \max(T(x), I(x), F(x)) > 1\}.$$

Definition 2.25 (Neutrosophic Underset)**.** [81–83] Let U be a universe of discourse, and let $T(x), I(x), F(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively. For a neutrosophic underset A, these functions satisfy:

$$T(x), I(x), F(x) : U \to [\Psi, 1], \quad \Psi < 0.$$

A Neutrosophic Underset is given by:

$$A = \{(x, T(x), I(x), F(x)) \mid x \in U, \exists x \in U \text{ such that } \min(T(x), I(x), F(x)) < 0\}.$$

Definition 2.26 (Neutrosophic Offset)**.** [81–83] Let U be a universe of discourse, and let $T(x), I(x), F(x)$ represent the truth, indeterminacy, and falsity membership functions, respectively. For a neutrosophic offset A, these functions satisfy:

$$T(x), I(x), F(x) : U \to [\Psi, \Omega], \quad \Psi < 0, \quad \Omega > 1.$$

A Neutrosophic Offset is given by:

$$A = \{(x, T(x), I(x), F(x)) \mid x \in U, \exists x \in U \text{ such that } \min(T(x), I(x), F(x)) < 0 \text{ and } \max(T(x), I(x), F(x)) > 1\}.$$

Definition 2.27 (HyperNeutrosophic Overset/Underset/Offset)**.** Let U be a universe of discourse, and let $\Psi < 0 < 1 < \Omega$. Define intervals:

$$\text{(Overset domain): } [0,\Omega]^3, \quad \text{(Underset domain): } [\Psi,1]^3, \quad \text{(Offset domain): } [\Psi,\Omega]^3.$$

A *HyperNeutrosophic Overset* (HNO) $\widetilde{A}$, *HyperNeutrosophic Underset* (HNU) $\widetilde{B}$, or *HyperNeutrosophic Offset* (HNOf) $\widetilde{C}$ is a mapping:

$$\widetilde{A} : U \to \mathcal{P}([0,\Omega]^3), \quad \widetilde{B} : U \to \mathcal{P}([\Psi,1]^3), \quad \widetilde{C} : U \to \mathcal{P}([\Psi,\Omega]^3),$$

respectively, such that:

- *(Overset case):* There exists at least one $x \in U$ for which some $(T,I,F) \in \widetilde{A}(x)$ satisfies $\max\{T,I,F\} > 1$.
- *(Underset case):* There exists at least one $x \in U$ for which some $(T,I,F) \in \widetilde{B}(x)$ satisfies $\min\{T,I,F\} < 0$.
- *(Offset case):* There exists at least one $x \in U$ for which some $(T,I,F) \in \widetilde{C}(x)$ satisfies $\min\{T,I,F\} < 0$ and $\max\{T,I,F\} > 1$.

Hence, each element x is assigned a *set* of membership triples, possibly extending below 0 or above 1, depending on overset, underset, or offset definitions.

Theorem 2.28. *Every Neutrosophic Overset is a special case of a HyperNeutrosophic Overset.*

Proof. A *Neutrosophic Overset* A on U associates each $x \in U$ with one triple $(T(x), I(x), F(x))$ where $\max(T(x), I(x), F(x)) > 1$ for at least one x. In Definition 2.27, a *HyperNeutrosophic Overset* $\widetilde{A}$ maps $x \in U$ to a *set* of $(T,I,F) \in [0,\Omega]^3$. We embed A by letting

$$\widetilde{A}(x) = \Big\{\big(T(x), I(x), F(x)\big)\Big\}$$

(a singleton). Thus, each element is assigned exactly one triple. The overset condition $\max\{T(x), I(x), F(x)\} > 1$ for some x remains, so A is recovered exactly as a degenerate (single membership) hyperneutrosophic overset. □

Theorem 2.29. *Every HyperNeutrosophic Set is a special case of a HyperNeutrosophic Overset (resp. Underset, Offset) by restricting Ω to 1 (resp. Ψ to 0, $\Psi = 0, \Omega = 1$).*

Proof. A *HyperNeutrosophic Set* $\tilde{A}$ uses $[0,1]^3$ for memberships. In the overset domain we have $[0,\Omega]^3$, with $\Omega > 1$. If we take $\Omega = 1$, that domain reverts to $[0,1]^3$, so $\widetilde{A}$ is embedded trivially. The same logic applies to underset (set $\Psi = 0$) or offset (set $\Psi = 0, \Omega = 1$). Hence, ignoring the extended domain merges the set back into $[0,1]^3$. □

Theorem 2.30. *Every Neutrosophic Underset/Offset is a special case of a HyperNeutrosophic Underset/Offset, respectively.*

Proof. Parallel to Theorem 2.28, but for underset/offset. For an underset, we let $\widetilde{B}(x) = \{(T(x), I(x), F(x))\}$, a singleton in $[\Psi,1]^3$, with $\min\{T,I,F\} < 0$ for at least one x. The offset proof is similar: $\widetilde{C}(x) = \{(T,I,F)\}$ with $\min < 0$ and $\max > 1$ for at least one x. Hence, singletons in the hyper domain replicate the single-valued case. □

Definition 2.31 (n-SuperHyperNeutrosophic Overset/Underset/Offset)**.** Let U be a universe, and let $\Psi < 0 < 1 < \Omega$. For each $n \geq 1$, define $\mathcal{P}_n(U)$ as the n-th nested power set of U, and consider

$$\mathcal{P}_n([0,\Omega]^3), \quad \mathcal{P}_n([\Psi,1]^3), \quad \mathcal{P}_n([\Psi,\Omega]^3)$$

for the overset, underset, and offset domains, respectively. Then:

- An *n-SuperHyperNeutrosophic Overset* $\widetilde{A}_n$ is a mapping

$$\widetilde{A}_n : \mathcal{P}_n(U) \to \mathcal{P}_n\big([0, \Omega]^3\big),$$

  with at least one $A \in \mathcal{P}_n(U)$ and some triple $(T, I, F) \in \widetilde{A}_n(A)$ such that $\max(T, I, F) > 1$.

- An *n-SuperHyperNeutrosophic Underset* $\widetilde{B}_n$ uses $\mathcal{P}_n([\Psi, 1]^3)$ with at least one $A \in \mathcal{P}_n(U)$ and some (T, I, F) where $\min(T, I, F) < 0$.

- An *n-SuperHyperNeutrosophic Offset* $\widetilde{C}_n$ uses $\mathcal{P}_n([\Psi, \Omega]^3)$ with at least one $A \in \mathcal{P}_n(U)$ and (T, I, F) where $\min(T, I, F) < 0$ and $\max(T, I, F) > 1$.

Thus, each n-th level subset is assigned a *set* of membership triples in the extended domain, capturing overset, underset, or offset behavior in an n-superhyper environment.

Theorem 2.32. *Every HyperNeutrosophic Overset (Underset, Offset) is a special case of an n-SuperHyperNeutrosophic Overset (Underset, Offset) for* $n = 1$.

Proof. Take the overset case for illustration (similar for underset/offset). Let $\widetilde{A}$ be a HyperNeutrosophic Overset mapping $U \to \mathcal{P}([0, \Omega]^3)$. In the n-super version, for $n = 1$ we have:

$$\widetilde{A}_1 : \mathcal{P}_1(U) = \mathcal{P}(U) \to \mathcal{P}_1\big([0, \Omega]^3\big) = \mathcal{P}([0, \Omega]^3).$$

Define $\widetilde{A}_1(\{x\}) := \widetilde{A}(x)$ and $\widetilde{A}_1(A) = \emptyset$ for $A \neq \{x\}$. Then singletons in $\mathcal{P}(U)$ recover exactly the membership sets $\widetilde{A}(x)$. The overset condition remains ($\max(T, I, F) > 1$ for some triple). Similarly for underset/offset. Hence, each HyperNeutrosophic overset/underset/offset is embedded in the n-super version with $n = 1$. □

Theorem 2.33. *Every Neutrosophic Overset (Underset, Offset) is a special case of an n-SuperHyperNeutrosophic Overset (Underset, Offset) by letting* $n = 1$ *and singletons.*

Proof. Parallels the logic in the previous theorems: we define $\widetilde{A}_n(\{x\}) = \{(T(x), I(x), F(x))\}$, a singleton, ensuring the overset/underset/offset condition is satisfied for at least one triple. This replicates the single-valued scenario in the n-superhyper context. □

Theorem 2.34. *Every n-SuperHyperNeutrosophic Set is a special case of an n-SuperHyperNeutrosophic Overset (Underset, Offset) by restricting* Ω *to* 1 *(resp.* $\Psi = 0$, $\Psi = 0, \Omega = 1$*).*

Proof. Same scaling or restriction arguments: if $\Omega = 1$ we lose the overset possibility above 1, if $\Psi = 0$ we lose negativity, etc. This recovers a normal n-SuperHyperNeutrosophic membership in $[0, 1]^3$. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[2] Muhammad Irfan Ali. Another view on q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 33:2139 – 2153, 2018.

[3] Mumtaz Ali, Irfan Deli, and Florentin Smarandache. The theory of neutrosophic cubic sets and their applications in pattern recognition. *Journal of intelligent & fuzzy systems*, 30(4):1957–1963, 2016.

[4] Rasha Almajed, Watson Thompson, et al. Improving the perfoamnce of fog-assisted internet of things networks using bipolar trapezoidal neutrosophic sets. *International Journal of Wireless & Ad Hoc Communication*, 6(1), 2023.

[5] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[6] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[7] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[8] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[9] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[10] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[11] Vivek Badhe, RS Thakur, and GS Thakur. Vague set theory for profit pattern and decision making in uncertain data. *International journal of advanced computer science and applications*, 6(6):58–64, 2015.

[12] Durga Banerjee, Bibhas C Giri, Surapati Pramanik, and Florentin Smarandache. Gra for multi attribute decision making in neutrosophic cubic set environment. *Neutrosophic Sets and Systems*, 15:60–69, 2017.

[13] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Shortest path problem under trapezoidal neutrosophic information. *Infinite Study*, 2017.

[14] Wen-Hua Cui and Jun Ye. Logarithmic similarity measure of dynamic neutrosophic cubic sets and its application in medical diagnosis. *Computers in Industry*, 111:198–206, 2019.

[15] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[16] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[17] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[18] Takaaki Fujita. Natural n-superhyper plithogenic language.

[19] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[20] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[21] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[22] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[23] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[24] Takaaki Fujita. Short communication of big plithogenic science and deep plithogenic science. 2024. preprint (researchgate).

[25] Takaaki Fujita. Short note of bunch graph in fuzzy, neutrosophic, and plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2024.

[26] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[27] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[28] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[29] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). July 2024.

[30] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[31] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[32] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[33] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[34] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[35] Takaaki Fujita. Short note of even-hole-graph for uncertain graph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 351, 2025.

[36] Takaaki Fujita. Short note of extended hyperplithogenic sets, 2025. Preprint.

[37] Takaaki Fujita. A short note on the basic graph construction algorithm for plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 274, 2025.

[38] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[39] Takaaki Fujita. Some types of hyperneutrosophic set (2): Complex, single-valued triangular, fermatean, and linguistic sets. *Preprint*, 2025.

[40] Takaaki Fujita. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. 2025.

[41] Takaaki Fujita and Florentin Smarandache. A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 308.

[42] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[43] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[44] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[45] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[46] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[47] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[48] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[49] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[50] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[51] Muhammad Gulistan, Ahmed Elmoasry, and Naveed Yaqoob. N-version of the neutrosophic cubic set: application in the negative influences of internet. *The Journal of Supercomputing*, 77(10):11410–11431, 2021.

[52] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[53] Hariwan Z Ibrahim. Multi-attribute group decision-making based on bipolar n, m-rung orthopair fuzzy sets. *Granular Computing*, 8(6):1819–1836, 2023.

[54] Hariwan Z Ibrahim. Multi-criteria decision-making based on similarity measures on interval-valued bipolar n, m-rung orthopair fuzzy sets. *Granular Computing*, 9(1):5, 2024.

[55] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[56] Young Bae Jun, Florentin Smarandache, and Chang Su Kim. Neutrosophic cubic sets. *New mathematics and natural computation*, 13(01):41–54, 2017.

[57] Hüseyin Kamacı, Harish Garg, and Subramanian Petchimuthu. Bipolar trapezoidal neutrosophic sets and their dombi operators with applications in multicriteria decision making. *Soft Computing*, 25(13):8417–8440, 2021.

[58] Muhammad Jabir Khan, Poom Kumam, and Meshal Shutaywi. Knowledge measure for the q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 36(2):628–655, 2021.

[59] Kiran Khatter. Interval valued trapezoidal neutrosophic set: multi-attribute decision making for prioritization of non-functional requirements. *Journal of Ambient Intelligence and Humanized Computing*, 12(1):1039–1055, 2021.

[60] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[61] BongJu Lee and Yong Sik Yun. The generalized trapezoidal fuzzy sets. *Journal of the chungcheong mathematical society*, 24(2):253–266, 2011.

[62] Donghai Liu, Xiao hong Chen, and Dan Peng. Some cosine similarity measures and distance measures between q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34:1572 – 1587, 2019.

[63] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[64] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[65] Nivetha Martin, Priya Priya.R, and Florentin Smarandache. Decision making on teachers' adaptation to cybergogy in saturated interval- valued refined neutrosophic overset /underset /offset environment. *International Journal of Neutrosophic Science*, 2020.

[66] Jerry M Mendel. On computing the similarity of trapezoidal fuzzy sets using an automated area method. *Information Sciences*, 589:716–737, 2022.

[67] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[68] p. geetha and K. Anitha. Single valued neutrosophic soft over / under / offsets. 2016.

[69] Xindong Peng and Lin Liu. Information measures for q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 34:1795 – 1834, 2019.

[70] Surapati Pramanik, Shyamal Dalapati, Shariful Alam, and Tapan Kumar Roy. Nc-todim-based magdm under a neutrosophic cubic set environment. *Information*, 8(4):149, 2017.

[71] Pratibha Rani and Arunodaya Raj Mishra. Multi-criteria weighted aggregated sum product assessment framework for fuel technology selection using q-rung orthopair fuzzy sets. *Sustainable Production and Consumption*, 24:90–104, 2020.

[72] Vanita Rani and Satish Kumar. An innovative distance measure for quantifying the dissimilarity between q-rung orthopair fuzzy sets. *Decision Analytics Journal*, 11:100440, 2024.

[73] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[74] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[75] S Santhoshkumar, J Aldring, and D Ajay. Analyzing aggregation operators on complex q-rung orthopair neutrosophic sets with their application. In *International Conference on Intelligent and Fuzzy Systems*, pages 744–751. Springer, 2024.

[76] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[77] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[78] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[79] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[80] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[81] Florentin Smarandache. Degrees of membership¿ 1 and¡ 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[82] Florentin Smarandache. Neutrosophic overset. *Neutrosophic Underset, and Neutrosophic Offset: Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics, Pons Editions Brussels*, 2016.

[83] Florentin Smarandache. Neutrosophic overset applied in physics. In *69th Annual Gaseous Electronics Conference, Bochum, Germany [through American Physical Society (APS)]*, 2016.

[84] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[85] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[86] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[87] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[88] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[89] Florentin Smarandache. *Practical Applications of the Independent Neutrosophic Components and of the Neutrosophic Offset Components*. Infinite Study, 2021.

[90] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[91] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[92] Florentin Smarandache. Operators for uncertain over/under/off-sets/-logics/probabilities/-statistics. *Neutrosophic Sets and Systems*, 79:493–500, 2025.

[93] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[94] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[95] Florentin Smarandache, Miguel A. Quiroz-Martínez, Jesús Estupiñán, Noel Batista Hernández, and Maikel Yelandi Leyva Vazquez. Application of neutrosophic offsets for digital image processing. 2020.

[96] Metawee Songsaeng, Aiyared Iampan, et al. *Neutrosophic cubic set theory applied to UP-algebras*. PhD thesis, University of Phayao, 2024.

[97] Samajh Singh Thakur and Saeid Jafari. (m, a, n)-fuzzy neutrosophic sets and their topological structure. *Neutrosophic Sets and Systems*, 73:592–605, 2024.

[98] Michael Gr Voskoglou, Florentin Smarandache, and Mona Mohamed. *q-Rung Neutrosophic Sets and Topological Spaces*. Infinite Study, 2024.

[99] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[100] Zhi Xiao, Sisi Xia, Ke Gong, and Dan Li. The trapezoidal fuzzy soft set and its application in mcdm. *Applied Mathematical Modelling*, 36:5844–5855, 2012.

[101] Jun Ye. Trapezoidal neutrosophic set and its application to multiple attribute decision-making. *Neural computing and Applications*, 26:1157–1166, 2015.

[102] P Yiarayong. Some weighted aggregation operators of quadripartitioned single-valued trapezoidal neutrosophic sets and their multi-criteria group decision-making method for developing green supplier selection criteria. *OPSEARCH*, pages 1–55, 2024.

[103] YS Yun. Parametric operations for two 2-dimensional trapezoidal fuzzy sets. *Journal of Algebra & Applied Mathematics*, 18(1), 2020.

[104] YS Yun. Parametric operations between 3-dimensional triangular fuzzy number and trapezoidal fuzzy set. *Journal of Algebra & Applied Mathematics*, 21(2), 2023.

[105] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[106] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[107] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[108] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[109] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[110] Jianming Zhan, Madad Khan, Muhammad Gulistan, and Ahmed Ali. Applications of neutrosophic cubic sets in multi-criteria decision-making. *International Journal for Uncertainty Quantification*, 7(5), 2017.

Chapter 7
Some Types of HyperNeutrosophic Set (5): Support, Paraconsistent, Faillibilist, and Others

Takaaki Fujita [1] * *and Florentin Smarandache*[2],
[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp
[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

This paper builds upon the foundational advancements introduced in [14, 25–27]. The Neutrosophic Set offers a versatile mathematical framework for addressing uncertainty through its three membership functions: truth, indeterminacy, and falsity. Extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set have been recently proposed to tackle increasingly sophisticated and multidimensional problems. Detailed formal definitions of these concepts can be found in [20].

In this paper, we extend various specialized classes of Neutrosophic Sets—namely, the Support Neutrosophic Set, Neutrosophic Intuitionistic Set (distinct from the Intuitionistic Fuzzy Set), Neutrosophic Paraconsistent Set, Neutrosophic Faillibilist Set, Neutrosophic Paradoxist Set, Neutrosophic Pseudo-Paradoxist Set, Neutrosophic Tautological Set, Neutrosophic Nihilist Set, Neutrosophic Dialetheist Set, and Neutrosophic Trivialist Set—by utilizing the frameworks of the Hyperneutrosophic Set and the SuperHyperneutrosophic Set.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. The analysis utilizes classical set-theoretic operations and extends them into advanced frameworks. For readers seeking a deeper understanding of foundational set theory, resources such as [10, 36, 37, 41] are recommended. Additionally, the referenced literature offers a comprehensive exploration of the principles and applications of Neutrosophic Sets.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To address uncertainty, vagueness, and imprecision in decision-making processes, numerous set-theoretic frameworks have been developed. These frameworks include Fuzzy Sets, which were introduced in foundational works such as those by Zadeh [62–70]. Another prominent framework is Intuitionistic Fuzzy Sets, extensively studied by Atanassov and others [2–7]. Vague Sets, introduced and developed by researchers, also contribute significantly to this domain [1, 8, 34, 43, 47]. Furthermore, the Hyperfuzzy Set is known as one of the extended concepts of the Fuzzy Set [9, 13, 22, 23, 35, 38–40, 42, 44, 60].

Neutrosophic Sets, first introduced by Smarandache, offer a powerful means of capturing indeterminacy, allowing for more nuanced decision-making models [16–18,24,28–33,51,52,58]. Neutrosophic Sets generalize Fuzzy Sets by introducing an additional component: indeterminacy, alongside truth and falsity [49–52]. This enhancement allows for a richer and more precise representation of uncertainty and ambiguity.

To address increasingly complex scenarios, HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets have been developed. These advanced models are particularly suited for high-dimensional and intricate problem spaces [15, 20, 55]. Relevant definitions and simple examples are provided below.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset). [19,46] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset). (cf. [11,19,21,48,56])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Neutrosophic Set). [51,52] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.5 (Neutrosophic Set). *Scenario:* Assessing public opinion on a controversial policy.

Example: Let $X = \{\text{Alice}, \text{Bob}, \text{Charlie}\}$, representing individuals with varying opinions on the policy. The membership functions represent their support (T), uncertainty (I), and opposition (F) as follows:

- For Alice: $T_A(\text{Alice}) = 0.8$ (80% support), $I_A(\text{Alice}) = 0.1$ (10% uncertain), $F_A(\text{Alice}) = 0.1$ (10% oppose).
- For Bob: $T_A(\text{Bob}) = 0.5$, $I_A(\text{Bob}) = 0.3$, $F_A(\text{Bob}) = 0.2$.
- For Charlie: $T_A(\text{Charlie}) = 0.3$, $I_A(\text{Charlie}) = 0.4$, $F_A(\text{Charlie}) = 0.3$.

This representation allows nuanced analysis, reflecting both certainty and uncertainty in opinions.

Definition 1.6 (HyperNeutrosophic Set). (cf. [12,15,20,22,55]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 1.7 (HyperNeutrosophic Set). *Scenario:* Analyzing customer satisfaction for multiple products, considering evaluations from different dimensions or individuals.

Example: Let $X = \{\text{Product A}, \text{Product B}\}$, where each product has multi-dimensional satisfaction scores represented by sets of neutrosophic triplets:

- For Product A:
$$\tilde{\mu}(\text{Product A}) = \{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1)\},$$
representing two customers' evaluations where each triplet denotes degrees of truth, indeterminacy, and falsity.

- For Product B:
$$\tilde{\mu}(\text{Product B}) = \{(0.6, 0.3, 0.1), (0.5, 0.4, 0.1)\}.$$

This structure enables richer analysis by aggregating diverse customer feedback for a comprehensive view.

Definition 1.8 (n-SuperHyperNeutrosophic Set)**.** (cf. [12, 15, 20, 22]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,
$$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$
representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 1.9 (n-SuperHyperNeutrosophic Set)**.** *Scenario:* Multi-level hierarchical analysis of climate change impacts.

Example: Let $X = \{\text{Temperature, Rainfall, Sea Level}\}$, representing key factors influenced by climate change. We consider a three-level hierarchy:

- *Level 1:* Regions {Region 1, Region 2}.
- *Level 2:* Countries within regions, e.g., {Country A, Country B, Country C}.
- *Level 3:* Cities within countries, e.g., {City X, City Y, City Z}.

For each level, the n-SuperHyperNeutrosophic Set assigns a family of subsets with membership triplets. For instance:
$$\tilde{A}_3(\text{City X}) = \{(0.8, 0.15, 0.05), (0.7, 0.2, 0.1)\},$$
where each triplet represents truth (T), indeterminacy (I), and falsity (F) degrees at the city level. This approach integrates uncertainty at regional, country, and city scales for holistic decision-making.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Support-Neutrosophic set

A Support-Neutrosophic Set extends neutrosophic sets by adding a support membership function, modeling truth, indeterminacy, falsity, and support degrees [45, 61].

Definition 2.1 (Support-Neutrosophic set). [61] Let U be a universal set. A *Support-Neutrosophic Set* (SNS) A on U is characterized by four membership functions:

$$A = \{(x, T_A(x), I_A(x), F_A(x), s_A(x)) \mid x \in U\},$$

where:

- $T_A(x)$ is the *truth-membership function*,
- $I_A(x)$ is the *indeterminacy-membership function*,
- $F_A(x)$ is the *falsity-membership function*,
- $s_A(x)$ is the *support-membership function*.

Each membership function satisfies:

$$T_A(x), I_A(x), F_A(x), s_A(x) \in [0, 1] \quad \text{for all } x \in U.$$

There is no restriction on the sum of $T_A(x), I_A(x), F_A(x)$, so:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3,$$

and:

$$0 \le s_A(x) \le 1.$$

Definition 2.2 (Support HyperNeutrosophic Set (SHNS)). Let X be a non-empty set. A *Support HyperNeutrosophic Set* $\widehat{A}$ on X is defined as a mapping

$$\widehat{\mu} \colon X \longrightarrow \mathcal{P}\big([0, 1]^4\big),$$

where $\mathcal{P}([0, 1]^4)$ is the family of all non-empty subsets of the 4-dimensional unit hypercube $[0, 1]^4$. For each $x \in X$, $\widehat{\mu}(x) \subseteq [0, 1]^4$ is a set of quadruples $\big(T, I, F, s\big)$, where

$$(T, I, F, s) \in [0, 1]^4,$$

subject to the following neutrosophic-like constraint on (T, I, F):

$$0 \le T + I + F \le 3,$$

and the additional *support* coordinate $s \in [0, 1]$ is unrestricted apart from lying in $[0, 1]$.

Hence, each $x \in X$ can have multiple possible quadruples $\big(T, I, F, s\big)$, each representing degrees of *truth*, *indeterminacy*, *falsity*, and *support*, respectively, in a hyper-collection manner.

Theorem 2.3. *Let $\widehat{A}$ be a Support HyperNeutrosophic Set. Then:*

1. *If for every $x \in X$, $\widehat{\mu}(x)$ is a singleton, i.e. $\widehat{\mu}(x) = \{\big(T, I, F, s\big)\}$, we recover a* Support Neutrosophic Set.
2. *If we exclude the support coordinate (or fix it as a constant), we recover a* HyperNeutrosophic Set.

Hence, the concept of a Support HyperNeutrosophic Set generalizes both the Support Neutrosophic Set and the HyperNeutrosophic Set.

Proof. *(1) Reduction to Support Neutrosophic Set:* When $\widehat{\mu}(x)$ is restricted to exactly one quadruple (T, I, F, s) per $x \in X$, we have a single 4-tuple for each x. This precisely matches the usual definition of an SNS (where each x has membership degrees $T_A(x)$, $I_A(x)$, $F_A(x)$, and $s_A(x) \in [0, 1]$ with $T_A(x) + I_A(x) + F_A(x) \leq 3$).

(2) Reduction to HyperNeutrosophic Set: If we fix $s = 0$ or $s = 1$ (or remove s altogether), then $\widehat{\mu}(x) \subseteq [0, 1]^3$ for each x, and we keep the condition $T + I + F \leq 3$. This is exactly the definition of a HyperNeutrosophic Set $\tilde{A}$.

Therefore, $\widehat{A}$ unifies both structures in a single framework, completing the proof. □

Definition 2.4 (Support n-SuperHyperNeutrosophic Set (n-SHNS with Support))**.** Let X be a non-empty set. An *Support* n-SuperHyperNeutrosophic Set (abbreviated $\widehat{A}_n$) is defined as a mapping

$$\widehat{A}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big([0, 1]^4\big),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, and for $k \geq 2$,
$$\mathcal{P}_k(X) = \mathcal{P}\big(\mathcal{P}_{k-1}(X)\big),$$
  representing nested families of non-empty subsets of X up to depth k.
- $\mathcal{P}_n([0, 1]^4)$ is defined similarly for subsets of the 4-dimensional unit hypercube $[0, 1]^4$.

For each $A \in \mathcal{P}_n(X)$ and each quadruple $(T, I, F, s) \in \widehat{A}_n(A)$, we require:

$$T, I, F, s \in [0, 1], \quad 0 \leq T + I + F \leq 3.$$

That is, the first three coordinates (T, I, F) represent truth, indeterminacy, and falsity degrees (with a neutrosophic constraint), and the fourth coordinate $s \in [0, 1]$ represents a *support* degree. The "n-superhyper" aspect means we interpret $\widehat{A}_n$ at successively deeper levels of subsets in $\mathcal{P}_n(X)$.

Theorem 2.5. *(Unification of Support HyperNeutrosophic Set and n-SuperHyperNeutrosophic Set)*

Let $\widehat{A}_n$ be a Support n-SuperHyperNeutrosophic Set as in Definition 2.4. Then:

1. *If $n = 1$, $\widehat{A}_1$ reduces to a* Support HyperNeutrosophic Set *(see Definition 2.2), where each element in $\mathcal{P}_1(X) = \mathcal{P}(X)$ is just a subset $A \subseteq X$, and $\widehat{A}_1(A) \subseteq [0, 1]^4$.*
2. *If we remove the extra support coordinate from Definition 2.4, we recover the standard n-*SuperHyperNeutrosophic Set.

Thus, a Support n-SuperHyperNeutrosophic Set generalizes both the Support HyperNeutrosophic Set (when $n = 1$) and the n-SuperHyperNeutrosophic Set (when we remove or fix the support dimension).

Proof. *(1) Case $n = 1$:* By Definition 2.4, for $n = 1$ we map each $A \in \mathcal{P}_1(X) = \mathcal{P}(X)$ to a subset $\widehat{A}_1(A) \subseteq [0, 1]^4$. But each element of $\mathcal{P}(X)$ is just a subset of X. In practice, we can equate each $A \subseteq X$ with an element $x \in X$ if we want an element-wise perspective, obtaining precisely a "hyper-collection" of quadruples (T, I, F, s) for each x. That matches the structure of a Support HyperNeutrosophic Set.

(2) Removing the support dimension: If we ignore the fourth coordinate s, then each quadruple (T, I, F, s) reduces to $(T, I, F) \in [0, 1]^3$. The condition $T+I+F \leq 3$ yields exactly the standard n-SuperHyperNeutrosophic constraint. This proves that the new notion unifies both concepts in a single framework. □

2.2 Special Cases of Neutrosophic Sets

This subsection provides an explanation of the Special Cases of Neutrosophic Sets. The Neutrosophic Set can be transformed into various specialized set concepts, such as the Neutrosophic Intuitionistic Set (distinct from the Intuitionistic Fuzzy Set), Neutrosophic Paraconsistent Set, Neutrosophic Faillibilist Set, Neutrosophic Paradoxist Set, Neutrosophic Pseudo-Paradoxist Set, Neutrosophic Tautological Set, Neutrosophic Nihilist Set, Neutrosophic Dialetheist Set, and Neutrosophic Trivialist Set (cf. [53, 54, 57, 59]).

These concepts can be generalized using the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set frameworks. Due to the extensive nature of the proofs, they are omitted in this paper. For further details, refer to relevant sources such as [20].

Definition 2.6 (Non-Standard Unit Interval)**.** Let

$$]-0, 1+[\; = \; \{x \mid -0 \le x \le 1+\}$$

be the so-called *non-standard unit interval*, which can include "infinitesimal" parts below 0 (denoted by -0) and possibly "infinite" or "beyond 1" parts above 1 (denoted by 1+). In various treatments, one may restrict to the standard interval $[0, 1]$. However, in the most general neutrosophic sense, membership degrees can lie in this broader range $]-0, 1+[$.

Remark 2.7. Throughout, for each element x in the universe U (or X, S, etc.), we associate three subsets (or sub-values) $T, I, F \subseteq]-0, 1+[$. Intuitively:

$$T = \text{(truth-degree subset)}, \quad I = \text{(indeterminacy-degree subset)}, \quad F = \text{(falsity-degree subset)}.$$

We often denote an element x by $x(T, I, F)$, signifying that x has partial membership characterized by (T, I, F).

The Neutrosophic Set is redefined using the non-standard unit interval. The definition is provided below.

Definition 2.8 (Neutrosophic Set (Using Non-standard unit interval))**.** [51, 52, 58] Let U be a universe of discourse, and let $M \subseteq U$. A *Neutrosophic Set M* is defined by assigning to each $x \in U$ an ordered triple $\big(T_x, I_x, F_x\big)$, where

$$T_x, I_x, F_x \; \subseteq \;]-0, 1+[\quad \text{(the non-standard unit interval)},$$

representing the *truth*, *indeterminacy*, and *falsity* percentages (or degrees) of x belonging to M. Concretely, we write:

$$x(T_x, I_x, F_x).$$

These three subsets must satisfy the broad neutrosophic condition that

$$-0 \;\le\; \inf(T_x) + \inf(I_x) + \inf(F_x) \quad \text{and} \quad \sup(T_x) + \sup(I_x) + \sup(F_x) \;\le\; 3+,$$

allowing the sum of (truth + indeterminacy + falsity) to be anywhere in the extended range up to 3+, and similarly down to -0.

In the special (standard) case where each of T_x, I_x, F_x is a singleton in $[0, 1]$, one recovers simpler forms such as fuzzy, intuitionistic fuzzy, or other sets. But the full neutrosophic set framework permits negative infinitesimals or values beyond 1, depending on the chosen non-standard extension.

This definition generalizes the classical set ($T = 1$, $I = 0$, $F = 0$), fuzzy set ($T \in [0, 1]$, $I = 0$, $F = 1 - T$), intuitionistic fuzzy set, paraconsistent set, and many others (see discussions below).

Definition 2.9 (Hyperneutrosophic General Form)**.** Let U be the universe of discourse. A *Hyperneutrosophic Set* $\mathcal{H}$ assigns to each element $x \in U$ a triple

$$\big(\mathcal{T}(x), \; \mathcal{I}(x), \; \mathcal{F}(x)\big),$$

where each of $\mathcal{T}(x)$, $\mathcal{I}(x)$, $\mathcal{F}(x)$ is a *hyper-collection* of membership degrees (or subsets of membership degrees) in the non-standard interval $]-0, 1+[$. Concretely, we might view

$$\mathcal{T}(x) \;\subseteq\; \mathcal{P}\big(]-0, 1+[\big), \quad \mathcal{I}(x) \;\subseteq\; \mathcal{P}\big(]-0, 1+[\big), \quad \mathcal{F}(x) \;\subseteq\; \mathcal{P}\big(]-0, 1+[\big).$$

(Or alternatively each could be a function from some index set Λ to $]-0, 1+[$.)

We then impose the neutrosophic constraints in a *hyper* sense: for all choices of $\tau \in \mathcal{T}(x), \iota \in \mathcal{I}(x), \varphi \in \mathcal{F}(x)$, one must satisfy the usual bounds

$$-0 \le \inf(\tau) + \inf(\iota) + \inf(\varphi), \quad \sup(\tau) + \sup(\iota) + \sup(\varphi) \le 3+,$$

plus whatever extra condition the special class imposes.

Notation: We denote such a set as $\mathcal{H}: x \mapsto (\mathcal{T}(x), \mathcal{I}(x), \mathcal{F}(x))$.

Remark 2.10. All the special classes (Intuitionistic, Paraconsistent, etc.) can be "hyperized" by requiring that each $\mathcal{T}(x)$, $\mathcal{I}(x)$, $\mathcal{F}(x)$ satisfies the relevant constraints. For example, a *HyperNeutrosophic-Intuitionistic Set* might demand $\sup(\tau) + \sup(\iota) + \sup(\varphi) < 1$ for all $\tau \in \mathcal{T}(x)$, $\iota \in \mathcal{I}(x)$, etc.

Definition 2.11 (n-SuperHyperneutrosophic General Form)**.** An *n-SuperHyperneutrosophic Set* $\mathcal{H}^{(n)}$ extends Definition 2.9 to *n-level* hyper-memberships. At level 1, we assign $(\mathcal{T}_1(x), \mathcal{I}_1(x), \mathcal{F}_1(x))$. For each element $\tau \in \mathcal{T}_1(x)$, we define a second-level triple $(\mathcal{T}_2(\tau), \mathcal{I}_2(\tau), \mathcal{F}_2(\tau))$, etc., up to level n. Each level enforces the standard or special neutrosophic constraints in a hyper sense. Symbolically,

$$\mathcal{H}^{(n)}(x) = (\mathcal{T}_1(x), \mathcal{I}_1(x), \mathcal{F}_1(x)),$$

$$\text{and for each } \tau \in \mathcal{T}_1(x), \quad (\mathcal{T}_2(\tau), \mathcal{I}_2(\tau), \mathcal{F}_2(\tau)),$$

$$\vdots$$

$$\text{Level } n \text{ similarly.}$$

All the specialized conditions (e.g., $\sup(T) + \sup(I) + \sup(F) < 1$, or $\inf(I) > 0$, etc.) must hold across *all relevant sublevels* to preserve the special-case property in the n-SuperHyper sense.

Remark 2.12. Just like Hyperneutrosophic Sets, any of the specialized classes (Intuitionistic, Paraconsistent, etc.) can be "n-superhyperized." For instance, an *n-SuperHyperNeutrosophic Intuitionistic Set* would impose sup of each triple's sum < 1 across all n-levels of membership data, and so on.

2.2.1 Neutrosophic Intuitionistic Set

A Neutrosophic Intuitionistic Set generalizes Intuitionistic Fuzzy Sets by requiring that the supremum values of truth, indeterminacy, and falsity strictly sum to less than 1. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.13 (Neutrosophic Intuitionistic Set)**.** [53] An *Neutrosophic Intuitionistic Set* is a special class of neutrosophic set in which each element $x(T, I, F)$ satisfies

$$\sup(T) + \sup(I) + \sup(F) < 1.$$

Hence the total "sum" of truth, indeterminacy, and falsity is strictly below 1 when we consider the supremum values. This models *incomplete* knowledge about membership: no matter how large you make truth, indeterminacy, or falsity, they cannot jointly reach 1.

By comparison, classical *intuitionistic fuzzy sets* (in the sense of Atanassov) typically require $T + F \le 1$, with an *indeterminacy* of $1 - (T + F)$. The above neutrosophic condition generalizes that notion using possibly non-standard intervals.

Definition 2.14 (HyperNeutrosophic-Intuitionistic Set)**.** A *HyperNeutrosophic Intuitionistic Set* is a hyperneutrosophic set $\mathcal{H}$ such that for every $x \in U$ and for all $\tau \in \mathcal{T}(x)$, $\iota \in \mathcal{I}(x)$, $\varphi \in \mathcal{F}(x)$, the sum of supremum values satisfies

$$\sup(\tau) + \sup(\iota) + \sup(\varphi) < 1.$$

We can similarly impose that the sum of $\inf(\tau) + \inf(\iota) + \inf(\varphi)$ remains below 1, depending on the exact formalism. This ensures that each sub-element in the hyper-collection respects the "incomplete membership" principle of the original intuitionistic concept, but now in a hyper sense.

Definition 2.15 (n-SuperHyperNeutrosophic Intuitionistic Set)**.** An *n-SuperHyperNeutrosophic Intuitionistic Set* extends Definition by enforcing the same $\sup(\tau) + \sup(\iota) + \sup(\varphi) < 1$ (or similar) constraint at *each* of the n levels of membership. In other words, at level 1, for each $\tau_1 \in \mathcal{T}_1(x)$, $\iota_1 \in \mathcal{I}_1(x)$, $\varphi_1 \in \mathcal{F}_1(x)$, we require

$$\sup(\tau_1) + \sup(\iota_1) + \sup(\varphi_1) < 1,$$

and for level 2, each $\tau_2 \in \mathcal{T}_2(\tau_1)$, $\iota_2 \in \mathcal{I}_2(\iota_1)$, etc., must also satisfy the same incomplete sum constraint, and so on up to level n. This hierarchical layering captures multi-stage or multi-dimensional incomplete knowledge.

2.2.2 Neutrosophic Paraconsistent Set

A Neutrosophic Paraconsistent Set captures overlapping information by requiring truth, indeterminacy, and falsity supremum values to exceed 1. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.16 (Neutrosophic Paraconsistent Set)**.** [53] A *Neutrosophic Paraconsistent Set* is a special class of neutrosophic set in which each element $x(T, I, F)$ satisfies

$$\sup(T) + \sup(I) + \sup(F) \ > \ 1.$$

In other words, the total supremum of (T, I, F) strictly exceeds 1, capturing *paraconsistent information*, where one can have overlapping degrees that go beyond what is normally considered a single unity. This is closely tied to paraconsistent logic, where contradictions can coexist without trivialization.

Definition 2.17 (HyperNeutrosophic Paraconsistent Set)**.** A *HyperNeutrosophic Paraconsistent Set* is a hyperneutrosophic set $\mathcal{H}$ where each sub-element triple (τ, ι, φ) satisfies

$$\sup(\tau) + \sup(\iota) + \sup(\varphi) \ > \ 1.$$

Equivalently, the total membership across truth, indeterminacy, and falsity exceeds 1 in each hyper-subset. This extends the classical paraconsistent property ($\sup(T) + \sup(I) + \sup(F) > 1$) to every layer in the hyper-collection.

Definition 2.18 (n-SuperHyperNeutrosophic Paraconsistent Set)**.** An *n-SuperHyperNeutrosophic Paraconsistent Set* enforces the paraconsistent condition at *each of the n membership levels*. That is, for any element $x \in U$, at level k (where $1 \leq k \leq n$), each triple $(\tau_k, \iota_k, \varphi_k)$ satisfies

$$\sup(\tau_k) + \sup(\iota_k) + \sup(\varphi_k) > 1.$$

This captures a multi-layer logic scenario where paraconsistent overlap is present through all nested or recursive membership steps.

2.2.3 Neutrosophic Faillibilist Set

A Neutrosophic Faillibilist Set ensures every element has a strictly positive lower bound of indeterminacy, capturing inherent uncertainty. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.19 (Neutrosophic Faillibilist Set)**.** [53] A *Neutrosophic Faillibilist Set* is a class of neutrosophic set in which every element $x(T, I, F)$ has

$$\inf(I) \ > \ 0.$$

This means each element has a strictly positive lower bound of *indeterminacy*. In effect, no element is fully known (there is always some irreducible uncertainty).

Definition 2.20 (HyperNeutrosophic Faillibilist Set)**.** A *HyperNeutrosophic Faillibilist Set* is a hyperneutrosophic set $\mathcal{H}$ such that, for every $x \in U$, each $\iota \in \mathcal{I}(x)$ satisfies

$$\inf(\iota) > 0.$$

Hence, all sub-members for indeterminacy remain strictly above 0, generalizing the classic $\inf(I) > 0$ requirement to the hyper context.

Definition 2.21 (n-SuperHyperNeutrosophic Faillibilist Set)**.** An *n-SuperHyperNeutrosophic Faillibilist Set* extends the above property to n levels: at level k, each ι_k (for $\iota_k \in \mathcal{I}_k$) must have $\inf(\iota_k) > 0$. This ensures a permanent positive minimal indeterminacy across all nested membership layers.

2.2.4 Neutrosophic Paradoxist Set

A *Neutrosophic Paradoxist Set* is a class of neutrosophic set in which every element $x(T, I, F)$ has the specific form $x(1,\ I,\ 1)$. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.22 (Neutrosophic Paradoxist Set)**.** [53] A *Neutrosophic Paradoxist Set* is a class of neutrosophic set in which every element $x(T, I, F)$ has the specific form

$$x(1,\ I,\ 1).$$

Interpreted literally, each element belongs 100% to the set *and* does not belong 100% at the same time. Formally, $\inf(T) \geq 1$ (or T includes 1) and $\inf(F) \geq 1$. The indeterminacy I can be anything, but typically $I \subseteq]-0, 1+[$ as usual. This embodies a *paradox*: total membership and total non-membership simultaneously.

Definition 2.23 (HyperNeutrosophic Paradoxist Set)**.** A *HyperNeutrosophic Paradoxist Set* is a hyperneutrosophic set $\mathcal{H}$ where every sub-element triple (τ, ι, φ) satisfies

$$\tau \text{ contains } 1, \quad \varphi \text{ contains } 1.$$

In other words, $\inf(\tau) \geq 1$ and $\inf(\varphi) \geq 1$. This enforces "complete membership" and "complete non-membership" simultaneously at the hyper level. The indeterminacy ι can vary.

Definition 2.24 (n-SuperHyperNeutrosophic Paradoxist Set)**.** An *n-SuperHyperNeutrosophic Paradoxist Set* applies the condition $\inf(\tau_k) \geq 1$ and $\inf(\varphi_k) \geq 1$ at each level $k = 1, \ldots, n$. Thus, from the first to the nth membership layer, every sub-triple is paradoxical (full membership and full non-membership).

2.2.5 Neutrosophic Pseudo-Paradoxist Set

A Neutrosophic Pseudo-Paradoxist Set is a neutrosophic set where elements exhibit "partially total" membership or non-membership: one dimension is fully determined (100%), while the other is partially defined. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.25 (Neutrosophic Pseudo-Paradoxist Set)**.** [53] A *Neutrosophic Pseudo-Paradoxist Set* is a class of neutrosophic set in which every element $x(T, I, F)$ satisfies one of the following forms:

$$x(1,\ I,\ F) \quad \text{with} \quad 0 < \inf(F) \leq \sup(F) < 1,$$
$$\text{or} \quad x(T,\ I,\ 1) \quad \text{with} \quad 0 < \inf(T) \leq \sup(T) < 1.$$

Hence we have "partially total" membership or non-membership combined with partial membership. Concretely:

- In the first form, the element belongs 100% ($\inf(T) \geq 1$) and also partially does not belong (some fraction $F \in (0, 1)$).
- In the second form, the element partially belongs $T \in (0, 1)$ but also does not belong 100%.

This generalizes the idea of a paradox, but not at the "full 1 and full 1" for membership and non-membership. Instead, membership is fully 1 in one dimension, while the other dimension is strictly between 0 and 1, or vice versa.

Definition 2.26 (HyperNeutrosophic Pseudo-Paradoxist Set)**.** A *HyperNeutrosophic Pseudo-Paradoxist Set* is a hyperneutrosophic set such that, for every $\tau \in \mathcal{T}(x)$, $\varphi \in \mathcal{F}(x)$, either

$$\inf(\tau) \geq 1 \quad \text{and} \quad 0 < \inf(\varphi) \leq \sup(\varphi) < 1$$

or

$$0 < \inf(\tau) \leq \sup(\tau) < 1 \quad \text{and} \quad \inf(\varphi) \geq 1$$

for each hyper-subset. This extends the "pseudo-paradoxical" partial membership and partial non-membership to all sub-levels in the hyper-collection.

Definition 2.27 (n-SuperHyperNeutrosophic Pseudo-Paradoxist Set)**.** An *n-SuperHyperNeutrosophic Pseudo-Paradoxist Set* repeats these partial conditions $\big(1,\ F \in (0,1)\big)$ or $\big(T \in (0,1),\ 1\big)$ at each membership level k. Concretely, if $\tau_k \in \mathcal{T}_k, \varphi_k \in \mathcal{F}_k$, then either $\inf(\tau_k) \geq 1$ and $\sup(\varphi_k) < 1$, or vice versa, for each level k.

2.2.6 Neutrosophic Tautological Set

A Neutrosophic Tautological Set is a neutrosophic set where every element absolutely belongs to the set (T¿1) with no indeterminacy or falsity. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.28 (Neutrosophic Tautological Set)**.** [53] A *Neutrosophic Tautological Set* is a class of neutrosophic set in which every element x has the form

$$x(1+, -0, -0),$$

meaning it *absolutely* belongs to the set in all possible worlds/scenarios. Symbolically, $\inf(T) \geq 1+$ (an extended value beyond 1), while $\sup(I) \leq -0$ and $\sup(F) \leq -0$, i.e. no indeterminacy and no falsity even in extended sense. This is a "universal truth" membership scenario, hence "tautological."

Definition 2.29 (HyperNeutrosophic Tautological Set)**.** A *HyperNeutrosophic Tautological Set* is one in which, for every $\tau \in \mathcal{T}(x)$, we have $\inf(\tau) \geq 1+$, and simultaneously $\inf(\iota) \leq -0$ and $\inf(\varphi) \leq -0$ for all ι, φ in $\mathcal{I}(x)$, $\mathcal{F}(x)$. Thus each sub-collection ensures absolute truth across the entire hyper-structure.

Definition 2.30 (n-SuperHyperNeutrosophic Tautological Set)**.** An *n-SuperHyperNeutrosophic Tautological Set* demands that, at each level k, the truth sub-collection has elements with $\inf(\tau_k) \geq 1+$ and the indeterminacy/falsity sub-collections remain ≤ -0. Thus, from level 1 to level n, we have absolute membership, with no possibility of partial or negative membership.

2.2.7 Neutrosophic Nihilist Set

A Neutrosophic Nihilist Set is a neutrosophic set where every element absolutely does not belong (F¿1) with no truth or indeterminacy. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.31 (Neutrosophic Nihilist Set)**.** [53] A *Neutrosophic Nihilist Set* is a class of neutrosophic set in which every element x has the form

$$x(-0, -0, 1+),$$

meaning it *absolutely does not* belong to the set in all possible worlds. Symbolically, $\inf(F) \geq 1+$ (falsity beyond 1), while $\sup(T) \leq -0$ and $\sup(I) \leq -0$, i.e. no truth and no indeterminacy. The empty set is a particular case of a nihilist set.

Definition 2.32 (HyperNeutrosophic Nihilist Set)**.** A *HyperNeutrosophic Nihilist Set* ensures $\inf(\varphi) \geq 1+$ for each $\varphi \in \mathcal{F}(x)$, while $\sup(\tau) \leq -0$ and $\sup(\iota) \leq -0$. In every hyper-subset, the element absolutely does not belong, across all sub-members.

Definition 2.33 (n-SuperHyperNeutrosophic Nihilist Set)**.** An *n-SuperHyperNeutrosophic Nihilist Set* repeats the $\inf(\varphi_k) \geq 1+$ condition at each of the n membership levels, ensuring total falsity and no truth/indeterminacy for all nested sub-layers.

2.2.8 Neutrosophic Dialetheist Set

A Neutrosophic Dialetheist Set allows elements to belong simultaneously to the set and its complement, modeling logical contradictions. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.34 (Neutrosophic Dialetheist Set)**.** [53] A *Neutrosophic Dialetheist Set* is a class of neutrosophic set that models a situation where some element(s) also belong to the complement of the set. Formally, there exists at least one element $x(T, I, F)$ in the set M such that x also belongs to the complement $C(M)$. Equivalently, there is an overlap between M and its complement for at least one x. In neutrosophic terms, one might express this by saying T for membership in M is non-zero (or high), and simultaneously T for membership in $C(M)$ is also non-zero. This is akin to dialetheism in logic, where contradictions can be true.

Definition 2.35 (HyperNeutrosophic Dialetheist Set)**.** A *HyperNeutrosophic Dialetheist Set* is a hyperneutrosophic set $\mathcal{H}$ where there exists at least one element $x \in U$ (and at least one sub-membership triple $\tau \in \mathcal{T}(x)$, etc.) that also belongs to the complement's hyper-sub-collection. In practice, this means at least one level of membership for x has $\tau > 0$ for both the set and its complement, reflecting the dialetheist notion that contradictory membership can be valid.

Definition 2.36 (n-SuperHyperNeutrosophic Dialetheist Set)**.** We say $\mathcal{H}^{(n)}$ is an *n-SuperHyperNeutrosophic Dialetheist Set* if, at some level(s) k, there is an element that belongs simultaneously to both $\mathcal{H}^{(n)}$ and its n-level complement. In other words, the contradiction is "allowed" or "realized" across (possibly multiple) sub-layers of membership data.

2.2.9 Neutrosophic Trivialist Set

A Neutrosophic Trivialist Set includes all its elements in both the set and its complement, representing universal contradictions. This concept can be extended using the frameworks of the Hyperneutrosophic Set and n-SuperHyperneutrosophic Set. The formal definition is provided below.

Definition 2.37 (Neutrosophic Trivialist Set)**.** [53] A *Neutrosophic Trivialist Set* is a class of neutrosophic set where every element also belongs to the complement. Formally, for every $x(T, I, F)$ in M, x is also in $C(M)$. That is, the intersection between M and $C(M)$ is not only non-empty, but actually contains all elements of M. In a classical sense, "everything is true" and "everything is false" at once. Trivialism is the position that all contradictions are in fact the case.

Definition 2.38 (HyperNeutrosophic Trivialist Set)**.** A *HyperNeutrosophic Trivialist Set* ensures that *every* element in M also belongs to the complement's hyper-collection. Formally, for every $x \in M$ and every triple $\tau \in \mathcal{T}(x)$, etc., there is a triple in the complement's hyper-collection that also certifies membership. This generalizes the trivialist idea that $\cap(M, C(M))$ is universal across all sub-members.

Definition 2.39 (n-SuperHyperNeutrosophic Trivialist Set)**.** An *n-SuperHyperNeutrosophic Trivialist Set* extends this universal overlap to all membership levels: from level 1 to level n, each sub-triple for any $x \in M$ is repeated in the membership structure of the complement, thus making everything "trivially" shared across all levels.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[2] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[3] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[4] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[5] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[6] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[7] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[8] Vivek Badhe, RS Thakur, and GS Thakur. Vague set theory for profit pattern and decision making in uncertain data. *International journal of advanced computer science and applications*, 6(6):58–64, 2015.

[9] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[10] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[11] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[12] Takaaki Fujita. Some types of hyperfuzzy set (2): Nonstationary, dual hesitant, interval-valued, and l-fuzzy set. *Preprint*.

[13] Takaaki Fujita. Some types of hyperfuzzy set: Bipolar, m-polar, q-rung orthopair, trapezoidal, linguistic, intuitionistic, picture, hesitant, spherical, type-m, offset, overset, and underset. *Preprint*.

[14] Takaaki Fujita. Some types of hyperneutrosophic set (4): Cubic, trapezoidal, q-rung orthopair, overset, underset, and offset.

[15] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[16] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[17] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[18] Takaaki Fujita. Short note of bunch graph in fuzzy, neutrosophic, and plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2024.

[19] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[20] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[21] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[22] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[23] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[24] Takaaki Fujita. Short note of even-hole-graph for uncertain graph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 351, 2025.

[25] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[26] Takaaki Fujita. Some types of hyperneutrosophic set (2): Complex, single-valued triangular, fermatean, and linguistic sets. *Preprint*, 2025.

[27] Takaaki Fujita. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. 2025.

[28] Takaaki Fujita and Florentin Smarandache. A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 308.

[29] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[30] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[31] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[32] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[33] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[34] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[35] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[36] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[37] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[38] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[39] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Length-fuzzy subalgebras in bck/bci-algebras. *Mathematics*, 6(1):11, 2018.

[40] Jeong Gi Kang and Hashem Bordbar. Further results on length fuzzy subalgebras of bck/bci-algebras. 2018.

[41] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[42] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[43] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[44] Zohreh Nazari and Batool Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16:173 – 185, 2018.

[45] M Parimala, M Karthika, Muhammad Riaz, Cenap Ozel, and Florentin Smarandache. Neutrosophic support rough digraph and its applications.

[46] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[47] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[48] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[49] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[50] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[51] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[52] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[53] Florentin Smarandache. *Definitions derived from neutrosophics*. Infinite Study, 2003.

[54] Florentin Smarandache. *A unifying field in logics: neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability: neutrsophic logic. Neutrosophy, neutrosophic set, neutrosophic probability*. Infinite Study, 2005.

[55] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[56] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[57] Florentin Smarandache. Three decades of neutrosophic and plithogenic theories with their applications (1995-2024). *Neutrosophic sets and systems*, 73(1):2, 2024.

[58] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[59] Florentin Smarandache and Maissam Jdid. An overview of neutrosophic and plithogenic theories and applications. 2023.

[60] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[61] Nguyen Xuan Thao, Florentin Smarandache, and Nguyen Van Dinh. *Support-neutrosophic set: a new concept in soft computing*. Infinite Study, 2017.

[62] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[63] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[64] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[65] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[66] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[67] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[68] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[69] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[70] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

Chapter 8
Some Types of HyperNeutrosophic Set (6): MultiNeutrosophic Set and Refined Neutrosophic Set

Takaaki Fujita [1] * *and Florentin Smarandache*[2],
[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp
[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

This paper builds on the foundational advancements introduced in [22, 29–32]. The Neutrosophic Set provides a flexible mathematical framework for managing uncertainty by utilizing three membership functions: truth, indeterminacy, and falsity. Recent extensions, such as the HyperNeutrosophic Set and the SuperHyperNeutrosophic Set, have been developed to address increasingly complex and multidimensional challenges. Comprehensive formal definitions of these concepts are provided in [26].

In this paper, we further extend various specialized classes of Neutrosophic Sets. Specifically, we explore extensions of the MultiNeutrosophic Set and the Refined Neutrosophic Set using HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, providing detailed analysis and examples.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section presents the foundational concepts and definitions necessary for the discussions in this paper.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

In addressing uncertainty, vagueness, and imprecision in decision-making, various set-theoretic models have been proposed. Among these, Fuzzy Sets introduced by Zadeh [60–66] provide a foundation for handling partial membership. Intuitionistic Fuzzy Sets, developed extensively by Atanassov [8–13], incorporate both membership and non-membership functions for better representation of uncertainty. Similarly, Vague Sets have been explored as a means to model imprecise data [1, 35, 40].

Hyperfuzzy Sets, a generalization of Fuzzy Sets, enable a broader representation of membership by considering subsets of the interval $[0, 1]$ [14, 21, 28, 38, 39]. These models provide enhanced flexibility in handling complex data scenarios.

Neutrosophic Sets, introduced by Smarandache, extend the Fuzzy Set framework by incorporating an indeterminacy component alongside truth and falsity [24, 33, 34, 47–50, 53, 57]. This approach allows for a richer characterization of uncertainty, making it particularly useful in complex decision-making contexts. Advanced studies have further refined Neutrosophic Sets, resulting in the development of HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets, which address high-dimensional and intricate problem domains [23, 26].

The following sections provide definitions and illustrative examples of these concepts, demonstrating their applicability and generalization potential.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [25, 44] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Example 1.3 (Powerset)**.** Let $S = \{a, b\}$. The powerset $\mathcal{P}(S)$ is the set of all subsets of S, including the empty set and S itself.

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Definition 1.4 (n-th Powerset)**.** (cf. [20, 25, 27, 46, 56])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Example 1.5 (First Iteration ($P_1(S)$))**.** By definition, $P_1(S) = \mathcal{P}(S)$. Therefore:

$$P_1(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

Example 1.6 (Second Powerset ($P_2(S)$))**.** The second powerset $P_2(S)$ is the powerset of $P_1(S)$. Since $P_1(S) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, we compute $\mathcal{P}(P_1(S))$:

$$P_2(S) = \mathcal{P}(P_1(S)) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \ldots, P_1(S)\}.$$

Example 1.7 (Third Powerset ($P_3(S)$))**.** The third powerset $P_3(S)$ is obtained by applying the powerset operation to $P_2(S)$:

$$P_3(S) = \mathcal{P}(P_2(S)).$$

Since $P_2(S)$ is a much larger set, $P_3(S)$ contains subsets of $P_2(S)$, including higher-order subsets.

Definition 1.8 (Neutrosophic Set)**.** [47, 48] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.9 (Neutrosophic Set in Customer Feedback Analysis)**.** *Scenario:* Evaluating customer satisfaction regarding a newly launched product (cf. [17]).

Example: Let $X = \{\text{Customer 1, Customer 2, Customer 3}\}$, representing a set of customers who provided feedback on the product. The membership functions represent their positive feedback (T), uncertainty (I), and negative feedback (F) as follows:

- For Customer 1: $T_A(\text{Customer 1}) = 0.7$ (70% positive feedback), $I_A(\text{Customer 1}) = 0.2$ (20% uncertainty), $F_A(\text{Customer 1}) = 0.1$ (10% negative feedback).
- For Customer 2: $T_A(\text{Customer 2}) = 0.6$, $I_A(\text{Customer 2}) = 0.1$, $F_A(\text{Customer 2}) = 0.3$.
- For Customer 3: $T_A(\text{Customer 3}) = 0.4$, $I_A(\text{Customer 3}) = 0.4$, $F_A(\text{Customer 3}) = 0.2$.

Interpretation:

- Customer 1 shows strong positive feedback with minimal uncertainty and negative feedback.
- Customer 2 provides moderate positive feedback but has a significant level of dissatisfaction ($F_A(\text{Customer 2}) = 0.3$).

- Customer 3 exhibits high uncertainty ($I_A(\text{Customer 3}) = 0.4$), indicating indecisiveness regarding their opinion of the product.

This example demonstrates how a Neutrosophic Set can be used to model customer feedback, capturing not only their satisfaction levels but also their uncertainties and dissatisfaction, enabling a more comprehensive understanding of customer opinions.

Definition 1.10 (HyperNeutrosophic Set)**.** (cf. [21,23,26,28,54]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS)* $\tilde{A}$ on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 1.11 (HyperNeutrosophic Set in Medical Diagnosis)**.** *Scenario:* Evaluating the health status of patients based on multiple diagnostic criteria, incorporating uncertainty and conflicting indicators (cf. [6, 15, 16, 19, 37, 41, 59]).

Let $X = \{\text{Patient 1}, \text{Patient 2}\}$, where each patient is assessed using a set of neutrosophic triplets representing degrees of health status (T), uncertainty in diagnosis (I), and severity of illness (F). The evaluations from multiple doctors or diagnostic tests are aggregated as follows:

- For Patient 1:

$$\tilde{\mu}(\text{Patient 1}) = \{(0.9, 0.05, 0.05), (0.8, 0.1, 0.1)\},$$

  indicating that one evaluation suggests high health status ($T = 0.9$) with minimal uncertainty ($I = 0.05$), while another is slightly less confident ($T = 0.8, I = 0.1$).

- For Patient 2:

$$\tilde{\mu}(\text{Patient 2}) = \{(0.4, 0.4, 0.2), (0.3, 0.5, 0.2)\},$$

  reflecting more uncertainty in diagnosis ($I = 0.4, 0.5$) and lower health status ($T = 0.4, 0.3$).

Interpretation:

- For Patient 1, the aggregated evaluations suggest a strong likelihood of good health, with very low uncertainty and minimal severity of illness.
- For Patient 2, the evaluations highlight significant uncertainty in the diagnosis, coupled with moderately low health status and moderate severity of illness.

This use of HyperNeutrosophic Sets allows for nuanced analysis in medical diagnosis, accommodating varying degrees of confidence and conflicting information from different sources or diagnostic tools.

Definition 1.12 (n-SuperHyperNeutrosophic Set)**.** (cf. [21, 23, 26, 28]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

$$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.

- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 1.13 (n-SuperHyperNeutrosophic Set)**.** *Scenario:* Multi-tiered analysis of supply chain reliability and uncertainty in a global logistics network.

Example: Let $X = \{\text{Production, Transportation, Warehousing}\}$, representing key components of a global supply chain. We consider a four-level hierarchy:

- *Level 1:* Global regions, e.g., {North America, Europe, Asia}.
- *Level 2:* Countries within regions, e.g., {USA, Germany, Japan}.
- *Level 3:* Distribution centers within countries, e.g., {Center A, Center B, Center C}.
- *Level 4:* Individual suppliers or transport hubs within distribution centers, e.g., {Supplier X, Supplier Y, Hub Z}.

For each level, the n-SuperHyperNeutrosophic Set assigns a family of subsets with neutrosophic membership triplets. For instance:

$$\tilde{A}_4(\text{Supplier X}) = \{(0.9, 0.05, 0.05), (0.85, 0.1, 0.05)\},$$

where each triplet represents:

- T: The reliability of the supplier, indicating truth in meeting delivery schedules.
- I: Uncertainty in performance due to external factors like weather or policy changes.
- F: The failure rate of the supplier in fulfilling commitments.

Interpretation:

- At the global region level, e.g., $\tilde{A}_1(\text{North America}) = \{(0.7, 0.2, 0.1), (0.75, 0.15, 0.1)\}$, reflects broader uncertainties like trade policies or labor strikes.
- At the country level, e.g., $\tilde{A}_2(\text{USA}) = \{(0.8, 0.1, 0.1), (0.85, 0.05, 0.1)\}$, captures national factors like infrastructure reliability or local regulations.
- At the distribution center level, e.g., $\tilde{A}_3(\text{Center A}) = \{(0.85, 0.1, 0.05), (0.8, 0.15, 0.05)\}$, focuses on facility-specific risks.

This hierarchical approach enables granular and comprehensive evaluation of supply chain reliability, accounting for uncertainty and risk at multiple levels of the network.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 m-Valued Refined Neutrosophic set

An m-Valued Refined Neutrosophic Set assigns m-refined truth, indeterminacy, and falsity membership values to elements, capturing granular uncertainty [4–6, 18, 36, 51, 52].

Definition 2.1 (m-Valued Refined Neutrosophic Set)**.** [51] An *m-Valued Refined Neutrosophic Set (m-VRNS)* is defined as follows:

Let U be a universe of discourse. An *m-Valued Refined Neutrosophic Set N* is represented as:

$$N = \{(x, \langle T_x, I_x, F_x \rangle) \mid x \in U\},$$

where:

- $T_x = \{T_x^1, T_x^2, \ldots, T_x^p\}$ is the set of refined truth-membership degrees for x,
- $I_x = \{I_x^1, I_x^2, \ldots, I_x^q\}$ is the set of refined indeterminacy-membership degrees for x,
- $F_x = \{F_x^1, F_x^2, \ldots, F_x^r\}$ is the set of refined falsity-membership degrees for x,
- $T_x^i, I_x^j, F_x^k \in [0, 1] \quad \forall i \in \{1, \ldots, p\}, j \in \{1, \ldots, q\}, k \in \{1, \ldots, r\}$,
- $p + q + r = m$, where m represents the total number of refined components.

The following condition must hold for each $x \in U$:

$$0 \leq \sum_{i=1}^{p} T_x^i + \sum_{j=1}^{q} I_x^j + \sum_{k=1}^{r} F_x^k \leq m.$$

Notes:

1. The sets T_x, I_x, F_x can represent different types of truth, indeterminacy, and falsity degrees (e.g., based on multiple criteria or perspectives).
2. The upper bound m ensures that the sum of all degrees (truth, indeterminacy, falsity) does not exceed the total refined capacity.
3. The model generalizes the classical neutrosophic set by allowing finer granularity in the representation of uncertainty.

Example 2.2 (m-Valued Refined Neutrosophic Set)**.** *Scenario:* Evaluating student performance in a multi-criteria assessment.

Let $U = \{\text{Alice, Bob, Charlie}\}$, where U represents students under evaluation. Each student is assessed on three main criteria: knowledge, teamwork, and creativity. The truth (T), indeterminacy (I), and falsity (F) degrees are refined into sub-components to represent their performance in finer granularity.

- For Alice:
$$T_{\text{Alice}} = \{0.8, 0.7\}, \quad I_{\text{Alice}} = \{0.1\}, \quad F_{\text{Alice}} = \{0.1, 0.2\},$$
  where:
  - T_{Alice}: High scores in knowledge (0.8) and teamwork (0.7).
  - I_{Alice}: Slight uncertainty (0.1) due to variable creativity.
  - F_{Alice}: Weak performance in advanced tasks (0.1) and group leadership (0.2).

  Total $m = 5$.

- For Bob:
$$T_{\text{Bob}} = \{0.6, 0.5\}, \quad I_{\text{Bob}} = \{0.2, 0.3\}, \quad F_{\text{Bob}} = \{0.4\},$$
where:
  - T_{Bob}: Moderate scores in creativity (0.6) and teamwork (0.5).
  - I_{Bob}: Higher uncertainty $(0.2, 0.3)$ due to inconsistent knowledge application.
  - F_{Bob}: Weak performance in collaboration (0.4).

  Total $m = 6$.
- For Charlie:
$$T_{\text{Charlie}} = \{0.7, 0.6\}, \quad I_{\text{Charlie}} = \{0.2\}, \quad F_{\text{Charlie}} = \{0.3, 0.2\},$$
where:
  - T_{Charlie}: Strong knowledge (0.7) and decent creativity (0.6).
  - I_{Charlie}: Moderate uncertainty (0.2) due to inconsistent teamwork.
  - F_{Charlie}: Weak performance in leadership (0.3) and practical implementation (0.2).

  Total $m = 5$.

Interpretation: Each student is evaluated on finer criteria, allowing for a nuanced assessment of their performance across different dimensions. This granularity is achieved using the m-Valued Refined Neutrosophic Set framework, where the truth, indeterminacy, and falsity degrees reflect detailed attributes under evaluation.

Definition 2.3 (m-Valued Refined HyperNeutrosophic Set (m-VRHNS))**.** Let U be a non-empty universe, and let
$$\mathcal{R}_m \;=\; \Big\{ (T, I, F) \;\Big|\; T, I, F \subseteq [0,1],\ |T| + |I| + |F| = m,\ \sum T + \sum I + \sum F \;\le\; m \Big\}$$
be the family of all *refined* triplets of sets representing truth, indeterminacy, and falsity degrees, subject to the total m constraint. Let $\widetilde{P}(\mathcal{R}_m)$ be the family of non-empty subsets of $\mathcal{R}_m$. A *m-Valued Refined HyperNeutrosophic Set (m-VRHNS)* $\widetilde{N}$ is a mapping
$$\widetilde{N} : U \;\longrightarrow\; \widetilde{P}(\mathcal{R}_m),$$
where for each $x \in U$, $\widetilde{N}(x) \subseteq \mathcal{R}_m$ is a *set* of refined triplets $(T_x^\alpha, I_x^\alpha, F_x^\alpha)$ with α indexing different possible combinations, each obeying
$$\sum(T_x^\alpha) + \sum(I_x^\alpha) + \sum(F_x^\alpha) \;\le\; m.$$

Interpretation:

- If $\widetilde{N}(x)$ is restricted to *one* triple (T_x, I_x, F_x), we get an *m-Valued Refined Neutrosophic Set* (Definition **??**).
- If $m = 3$ and each set is single-valued, we revert to a standard hyperneutrosophic set in $[0,1]^3$.

Theorem 2.4. *Every m-Valued Refined Neutrosophic Set is a special case of an m-Valued Refined HyperNeutrosophic Set.*

Proof. Let N be an m-Valued Refined Neutrosophic Set, with each $x \in U$ having one triple $(T_x, I_x, F_x) \in \mathcal{R}_m$. We define a mapping $\widetilde{N}$ by
$$\widetilde{N}(x) = \{\ (T_x, I_x, F_x)\ \}.$$
Hence each x has a singleton set of refined triplets. By construction, (T_x, I_x, F_x) obeys $\sum T_x + \sum I_x + \sum F_x \le m$. Therefore N is embedded into $\widetilde{N}$ as a degenerate (single triple) membership approach. □

Theorem 2.5. *By letting* $m = 3$ *and restricting each* T, I, F *to a single value in* $[0,1]$*, we revert to an ordinary HyperNeutrosophic Set in* $[0,1]^3$.

Proof. A *HyperNeutrosophic Set* $\tilde{\mu} : U \to \mathcal{P}([0,1]^3)$ assigns each x a set of triplets (T, I, F) with $T+I+F \le 3$. In the refined approach, $m = 3$ means we distribute the total 3 among T, I, F sets. If we force each set $T = \{t\}, I = \{i\}, F = \{f\}$, each containing exactly one numeric value, we get a single triplet $(t, i, f) \in [0,1]^3$, as usual. By allowing a set of such singletons, we reproduce the standard hyperneutrosophic membership in $[0,1]^3$. □

Definition 2.6 (m-Valued Refined n-SuperHyperNeutrosophic Set (m-VRHNS$_n$))**.** Let U be a non-empty universe, $n \ge 0$ an integer, and m a positive integer for refined neutrosophic components. Define $\mathcal{R}_m$ as in Definition 2.3 and $\widetilde{P}(\mathcal{R}_m)$ as the family of non-empty subsets of $\mathcal{R}_m$. Let $\widetilde{\mathcal{P}}_n^*(U)$ be the n-th power set hierarchy of U minus the empty set. An *m-Valued Refined n-SuperHyperNeutrosophic Set (m-VRHNS$_n$)* is a mapping:

$$\widetilde{N}_n : \widetilde{\mathcal{P}}_n^*(U) \longrightarrow \widetilde{P}(\mathcal{R}_m),$$

where each $A \in \widetilde{\mathcal{P}}_n^*(U)$ is assigned a *set* of refined neutrosophic triplets (T_A, I_A, F_A) with $|T_A|+|I_A|+|F_A| = m$, each satisfying

$$\sum T_A + \sum I_A + \sum F_A \le m.$$

Theorem 2.7. *Every m-Valued Refined HyperNeutrosophic Set is a special case of an m-Valued Refined n-SuperHyperNeutrosophic Set for* $n = 0$ *or* 1.

Proof. Let $\widetilde{N}$ be an m-Valued Refined HyperNeutrosophic Set: $U \to \widetilde{P}(\mathcal{R}_m)$. In Definition 2.6, if $n = 0$, $\widetilde{\mathcal{P}}_0^*(U) = U$, so

$$\widetilde{N}_0 : U \to \widetilde{P}(\mathcal{R}_m)$$

matches $\widetilde{N}$. Alternatively, if $n = 1$, define

$$\widetilde{N}_1(\{x\}) := \widetilde{N}(x), \quad \widetilde{N}_1(A) = \varnothing \text{ if } A \ne \{x\}.$$

Hence the single-level m-VRHNS is embedded in an n-super environment as a special case. □

Theorem 2.8. *By letting* $m = 3$ *and forcing each* (T_A, I_A, F_A) *to be singletons in* $[0,1]$*, we revert to a standard n-SuperHyperNeutrosophic set in numeric membership* $[0,1]^3$.

Proof. An *n-SuperHyperNeutrosophic Set* $\tilde{\mu}_n$ maps $\widetilde{\mathcal{P}}_n^*(U)$ to sets of triplets $(T, I, F) \in [0,1]^3$ with $T+I+F \le 3$. In the m-refined approach, $m = 3$ means we distribute the total across T, I, F. If each $T = \{t\}, I = \{i\}, F = \{f\}$ is a single numeric value, we replicate ordinary triplets. By allowing sets of such singletons, we replicate the standard n-SuperHyperNeutrosophic membership. □

2.2 MultiNeutrosophic Set

The concept of MultiNeutrosophic Sets has been extensively studied in several research papers [2, 3, 7, 55]. Related concepts, such as the Refined Fuzzy Set, are also well-documented in the literature [42, 43, 45, 58]. We extend the framework of MultiNeutrosophic Sets by incorporating HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets. The formal definitions and associated concepts are provided below.

Definition 2.9 (MultiNeutrosophic Set)**.** (cf. [55]) Let $\mathcal{U}$ be a universe of discourse, and let M be a subset of $\mathcal{U}$. A *MultiNeutrosophic Set (MNS) M* is defined as:

$$M = \left\{ \left(x, \langle T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \rangle\right) \;\middle|\; x \in \mathcal{U} \right\},$$

where:

- $p, r, s \ge 0$ with $p + r + s = n \ge 2$,
- At least one of p, r, s satisfies ≥ 2 to ensure the multiplicity of truth (T), indeterminacy (I), or falsehood (F),
- $T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s \subseteq [0,1]$,

- The following condition is satisfied:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Example 2.10 (Application of MultiNeutrosophic Set)**.** *Scenario:* Evaluating job applicants by multiple experts on their qualifications, experience, and cultural fit.

Example: Let $\mathcal{U} = \{\text{Alice, Bob, Charlie}\}$, representing the candidates. Each candidate is evaluated on three aspects by multiple experts:

- Truth (T): Positive qualifications and relevant skills,
- Indeterminacy (I): Ambiguity in the applicant's experience or behavior,
- Falsehood (F): Negative feedback or lack of required competencies.

The evaluations are provided by $n = 3 + 2 + 4 = 9$ sources as follows:

- For Alice:

$$T_{\text{Alice}} = \{0.8, 0.7, 0.6\}, \quad I_{\text{Alice}} = \{0.2, 0.3\}, \quad F_{\text{Alice}} = \{0.4, 0.3, 0.5, 0.2\}.$$

- For Bob:

$$T_{\text{Bob}} = \{0.7, 0.6, 0.5\}, \quad I_{\text{Bob}} = \{0.4, 0.3\}, \quad F_{\text{Bob}} = \{0.5, 0.4, 0.3, 0.2\}.$$

- For Charlie:

$$T_{\text{Charlie}} = \{0.5, 0.6, 0.4\}, \quad I_{\text{Charlie}} = \{0.1, 0.2\}, \quad F_{\text{Charlie}} = \{0.6, 0.5, 0.4, 0.3\}.$$

Each candidate is evaluated on multiple dimensions (truth, indeterminacy, and falsehood) by different experts. This structure enables nuanced decision-making by aggregating diverse perspectives. For instance, Alice's high truth values $(0.8, 0.7, 0.6)$ indicate strong qualifications, while her indeterminacy and falsehood values highlight specific areas of uncertainty or weakness.

Definition 2.11 (MultiHyperNeutrosophic Set)**.** Let $\mathcal{U}$ be a universe of discourse, and let M be a subset of $\mathcal{U}$. A *MultiHyperNeutrosophic Set (MHNS) M* is defined as:

$$M = \{(x, \tilde{\mu}(x)) \mid x \in \mathcal{U}\},$$

where $\tilde{\mu}(x) \subseteq \mathcal{P}([0,1]^3)$ is a set of neutrosophic membership triplets:

$$\tilde{\mu}(x) = \left\{(T_j, I_k, F_l) \;\middle|\; j = 1, \ldots, p;\ k = 1, \ldots, r;\ l = 1, \ldots, s\right\}.$$

The following conditions must be satisfied:

- $T_j, I_k, F_l \subseteq [0,1]$, where T_j, I_k, and F_l represent subsets of truth, indeterminacy, and falsehood degrees, respectively.
- The parameters $p, r, s \geq 1$ satisfy $p + r + s = n \geq 2$, ensuring at least one component has multiplicity ≥ 2.
- For all $x \in \mathcal{U}$, the following condition holds:

$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Example 2.12 (Example: MultiHyperNeutrosophic Set in Team Evaluation)**.** *Scenario:* A company is evaluating project teams based on performance, innovation, and collaboration. Feedback is provided by multiple experts.

Example: Let $\mathcal{U}$ = {Team A, Team B, Team C}, representing the project teams. For each team:

- Truth (T): Measures team performance, such as task completion and success rates.
- Indeterminacy (I): Reflects ambiguity in team processes or communication.
- Falsehood (F): Captures team failures or conflicts.

Evaluations by $p = 3$, $r = 2$, $s = 4$ sources are as follows:

- For Team A:
$$T_{\text{Team A}} = \{[0.8, 0.9], [0.7, 0.8], [0.6, 0.7]\},$$
$$I_{\text{Team A}} = \{[0.2, 0.3], [0.1, 0.2]\},$$
$$F_{\text{Team A}} = \{[0.1, 0.2], [0.3, 0.4], [0.2, 0.3], [0.4, 0.5]\}.$$
- For Team B:
$$T_{\text{Team B}} = \{[0.7, 0.8], [0.6, 0.7], [0.5, 0.6]\},$$
$$I_{\text{Team B}} = \{[0.3, 0.4], [0.2, 0.3]\},$$
$$F_{\text{Team B}} = \{[0.4, 0.5], [0.3, 0.4], [0.2, 0.3], [0.5, 0.6]\}.$$
- For Team C:
$$T_{\text{Team C}} = \{[0.6, 0.7], [0.5, 0.6], [0.4, 0.5]\},$$
$$I_{\text{Team C}} = \{[0.1, 0.2], [0.0, 0.1]\},$$
$$F_{\text{Team C}} = \{[0.5, 0.6], [0.4, 0.5], [0.3, 0.4], [0.6, 0.7]\}.$$

The evaluations provide a comprehensive view of each team's strengths (T), uncertainties (I), and weaknesses (F), enabling informed decision-making for resource allocation and performance improvement.

Definition 2.13 (Multi n-SuperHyperNeutrosophic Set)**.** Let $\mathcal{U}$ be a universe of discourse. A *Multi n-SuperHyperNeutrosophic Set (Multi n-SHNS)* is a recursive generalization of MultiHyperNeutrosophic Sets, defined as:
$$\tilde{M}_n : \mathcal{P}_n(\mathcal{U}) \to \mathcal{P}_n([0, 1]^3),$$
where:

- $\mathcal{P}_1(\mathcal{U}) = \mathcal{P}(\mathcal{U})$, the power set of $\mathcal{U}$,
- For $k \geq 2$,
$$\mathcal{P}_k(\mathcal{U}) = \mathcal{P}(\mathcal{P}_{k-1}(\mathcal{U})),$$
representing the k-th nested family of non-empty subsets of $\mathcal{U}$.
- For each $A \in \mathcal{P}_n(\mathcal{U})$, $\tilde{M}_n(A) \subseteq \mathcal{P}_n([0, 1]^3)$ is a family of subsets of neutrosophic triplets (T, I, F) satisfying:
$$0 \leq \sum_{j=1}^{p} \inf T_j + \sum_{k=1}^{r} \inf I_k + \sum_{l=1}^{s} \inf F_l \leq \sum_{j=1}^{p} \sup T_j + \sum_{k=1}^{r} \sup I_k + \sum_{l=1}^{s} \sup F_l \leq n.$$

Theorem 2.14. *A Multi n-SuperHyperNeutrosophic Set generalizes the concepts of MultiNeutrosophic Sets, HyperNeutrosophic Sets, and n-SuperHyperNeutrosophic Sets.*

Proof. 1. When $p = r = s = 1$, the structure reduces to a MultiNeutrosophic Set, as each evaluation is single-valued.

2. When $n = 1$, the structure aligns with the HyperNeutrosophic Set, mapping elements directly to neutrosophic triplets.

3. For $n > 1$, the recursive construction introduces hierarchical relationships, extending n-SuperHyperNeutrosophic Sets to multi-source contexts.

Thus, the Multi n-SuperHyperNeutrosophic Set encompasses all these frameworks. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[2] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, Ali HM Al-Obaidi, et al. New concepts in partner multineutrosophic topological space. *International Journal of Neutrosophic Science*, (3):172–72, 2024.

[3] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, Ali HM Al-Obaidi, et al. Partner sets for generalizations of multineutrosophic sets. *International Journal of Neutrosophic Science*, (1):08–8, 2024.

[4] Suriana Alias and Daud Mohamad. A review on neutrosophic set and its development. *Discovering Mathematics (Menemui Matematik)*, 39(2):61–69, 2017.

[5] Shawkat Alkhazaleh. n-valued refined neutrosophic soft set theory. *Journal of Intelligent & Fuzzy Systems*, 32(6):4311–4318, 2017.

[6] Shawkat Alkhazaleh and Ayman A Hazaymeh. N-valued refined neutrosophic soft sets and their applications in decision making problems and medical diagnosis. *Journal of Artificial Intelligence and Soft Computing Research*, 8(1):79–86, 2018.

[7] Kholood Mohammad Alsager. Decision-making framework based on multineutrosophic soft rough sets. *Mathematical Problems in Engineering*, 2022(1):2868970, 2022.

[8] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[9] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[10] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[11] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[12] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[13] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[14] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[15] Said Broumi, Irfan Deli, and Florentin Smarandache. N-valued interval neutrosophic sets and their application in medical diagnosis. *Critical Review, Center for Mathematics of Uncertainty, Creighton University, Omaha, NE, USA*, 10:45–69, 2015.

[16] Quang-Thinh Bui, My-Phuong Ngo, Vaclav Snasel, Witold Pedrycz, and Bay Vo. The sequence of neutrosophic soft sets and a decision-making problem in medical diagnosis. *International Journal of Fuzzy Systems*, 24:2036 – 2053, 2022.

[17] Sri Cempaka. *Integration Of Service Quality And Quality Function Deployment With Neutrosophic Ahp Methods In Customer Satisfaction Measurement Model (Case Study: PT Narmada Awet Muda Lombok)*. PhD thesis, Universitas Islam Indonesia, 2023.

[18] Jiqian Chen, Jun Ye, and Shigui Du. Vector similarity measures between refined simplified neutrosophic sets and their multiple attribute decision-making method. *Symmetry*, 9(8):153, 2017.

[19] Wen-Hua Cui and Jun Ye. Logarithmic similarity measure of dynamic neutrosophic cubic sets and its application in medical diagnosis. *Computers in Industry*, 111:198–206, 2019.

[20] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[21] Takaaki Fujita. Some types of hyperfuzzy set (2): Nonstationary, dual hesitant, interval-valued, and l-fuzzy set. *Preprint*.

[22] Takaaki Fujita. Some types of hyperneutrosophic set (4): Cubic, trapezoidal, q-rung orthopair, overset, underset, and offset.

[23] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[24] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[25] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[26] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[27] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[28] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[29] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[30] Takaaki Fujita. Some types of hyperneutrosophic set (2): Complex, single-valued triangular, fermatean, and linguistic sets. *Preprint*, 2025.

[31] Takaaki Fujita. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. 2025.

[32] Takaaki Fujita. Some types of hyperneutrosophic set (5): Support, paraconsistent, faillibilist, and others. 2025.

[33] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[34] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[35] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[36] Mohammad Hamidi, Arsham Borumand Saeid, and Florentin Smarandache. On neutrosophic refined eq–filters. *Journal of Intelligent & Fuzzy Systems*, 37(1):1181–1196, 2019.

[37] Masooma Raza Hashmi, Muhammad Riaz, and Florentin Smarandache. m-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis. *International Journal of Fuzzy Systems*, 22:273–292, 2020.

[38] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[39] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[40] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[41] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[42] Ambalal V Patel. Analytical structures and analysis of fuzzy pd controllers with multifuzzy sets having variable cross-point level. *Fuzzy sets and Systems*, 129(3):311–334, 2002.

[43] AMBALAL V Patel. Analytical structures of fuzzy pi controllers with multifuzzy sets under various t-norms, t-conorms and inference methods. *International Journal of Computational Cognition*, 1(3):93–141, 2003.

[44] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[45] Sabu Sebastian and TV Ramakrishnan. Multi-fuzzy sets: An extension of fuzzy sets. *Fuzzy Information and Engineering*, 3:35–43, 2011.

[46] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[47] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[48] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[49] Florentin Smarandache. *Definitions derived from neutrosophics*. Infinite Study, 2003.

[50] Florentin Smarandache. Neutrosophic physics: More problems, more solutions. 2010.

[51] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[52] Florentin Smarandache. Connections between extension logic and refined neutrosophic logic. *Neutrosophic Theory and Its Applications. Collected Papers*, 1:47–54, 2014.

[53] Florentin Smarandache. Degrees of membership¿ 1 and¡ 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[54] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[55] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[56] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[57] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[58] Apostolos Syropoulos. Fuzzy sets and fuzzy logic. In *Encyclopedia of Computer Science and Technology*, pages 459–468. CRC Press, 2017.

[59] Vakkas Ulucay, Adil Kılıç, Memet Sahin, and Harun Deniz. *A new hybrid distance-based similarity measure for refined neutrosophic sets and its application in medical diagnosis*. Infinite Study, 2019.

[60] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[61] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[62] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[63] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[64] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[65] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[66] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

Chapter 9

Some Types of HyperNeutrosophic Set (7): Type-m, Nonstationary, Subset-valued, and Complex Refined

Takaaki Fujita [1] * *and Florentin Smarandache*[2],

[1]* Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan. t171d603@gunma-u.ac.jp

[2] University of New Mexico, Gallup Campus, NM 87301, USA. smarand@unm.edu

Abstract

This paper builds upon the foundational advancements introduced in [26,39–43]. The Neutrosophic Set provides a versatile mathematical framework for addressing uncertainty through its three membership functions: truth, indeterminacy, and falsity [84]. Extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set have been recently proposed to address increasingly complex and multidimensional problems. Detailed formal definitions of these concepts can be found in [33].

In this paper, we extend the Type-m, Nonstationary, Subset-Valued, and Complex Refined Neutrosophic Sets using the Hyperneutrosophic Set and the SuperHyperneutrosophic Set frameworks.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set

1 Preliminaries and Definitions

This section provides an overview of the fundamental concepts and definitions essential for the discussions in this paper. The analysis utilizes classical set-theoretic operations and extends them into advanced frameworks. For readers seeking a deeper understanding of foundational set theory, resources such as [22, 56, 59, 65] are recommended. Additionally, the referenced literature offers a comprehensive exploration of the principles and applications of Neutrosophic Sets.

1.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

To address uncertainty, vagueness, and imprecision in decision-making processes, numerous set-theoretic frameworks have been developed. These frameworks include Fuzzy Sets, which were introduced in foundational works such as those by Zadeh [96–104]. Another prominent framework is Intuitionistic Fuzzy Sets, extensively studied by Atanassov and others [5–10]. Vague Sets, introduced and developed by researchers, also contribute significantly to this domain [3,11,52,68,79]. Furthermore, the Hyperfuzzy Set is known as one of the extended concepts of the Fuzzy Set [13,25,35,36,54,60–62,67,73,90].

Neutrosophic Sets, first introduced by Smarandache, offer a powerful means of capturing indeterminacy, allowing for more nuanced decision-making models [28, 29, 31, 38, 45–50, 83, 84, 89]. Neutrosophic Sets generalize Fuzzy Sets by introducing an additional component: indeterminacy, alongside truth and falsity [81–84]. This enhancement allows for a richer and more precise representation of uncertainty and ambiguity.

To address increasingly complex scenarios, HyperNeutrosophic Sets and n-SuperHyperNeutrosophic Sets have been developed. These advanced models are particularly suited for high-dimensional and intricate problem spaces [27, 33]. Relevant definitions and simple examples are provided below.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [32, 78] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (n-th Powerset)**.** (cf. [23,32,34,80,87])

The n-th powerset of a set H, denoted $P_n(H)$, is defined iteratively, starting with the standard powerset. The recursive construction is given by:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted $P_n^*(H)$, is defined recursively as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set removed.

Definition 1.4 (Neutrosophic Set)**.** [83, 84] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.5 (Neutrosophic Set)**.** *Scenario:* Assessing public opinion on a controversial policy.

Example: Let $X = \{\text{Alice, Bob, Charlie}\}$, representing individuals with varying opinions on the policy. The membership functions represent their support (T), uncertainty (I), and opposition (F) as follows:

- For Alice: $T_A(\text{Alice}) = 0.8$ (80% support), $I_A(\text{Alice}) = 0.1$ (10% uncertain), $F_A(\text{Alice}) = 0.1$ (10% oppose).
- For Bob: $T_A(\text{Bob}) = 0.5$, $I_A(\text{Bob}) = 0.3$, $F_A(\text{Bob}) = 0.2$.
- For Charlie: $T_A(\text{Charlie}) = 0.3$, $I_A(\text{Charlie}) = 0.4$, $F_A(\text{Charlie}) = 0.3$.

This representation allows nuanced analysis, reflecting both certainty and uncertainty in opinions.

Definition 1.6 (HyperNeutrosophic Set)**.** (cf. [24,27,33,35,85]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0,1]^3),$$

where $\mathcal{P}([0,1]^3)$ is the family of all non-empty subsets of the unit cube $[0,1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0,1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \leq T + I + F \leq 3.$$

Example 1.7 (HyperNeutrosophic Set)**.** *Scenario:* Analyzing customer satisfaction for multiple products, considering evaluations from different dimensions or individuals.

Example: Let $X = \{\text{Product A, Product B}\}$, where each product has multi-dimensional satisfaction scores represented by sets of neutrosophic triplets:

- For Product A:
$$\tilde{\mu}(\text{Product A}) = \{(0.8, 0.1, 0.1), (0.7, 0.2, 0.1)\},$$
representing two customers' evaluations where each triplet denotes degrees of truth, indeterminacy, and falsity.
- For Product B:
$$\tilde{\mu}(\text{Product B}) = \{(0.6, 0.3, 0.1), (0.5, 0.4, 0.1)\}.$$

This structure enables richer analysis by aggregating diverse customer feedback for a comprehensive view.

Definition 1.8 (n-SuperHyperNeutrosophic Set)**.** (cf. [24, 27, 33, 35]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0,1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \geq 2$,

  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$

  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0,1]^3)$ is defined similarly for the unit cube $[0,1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \leq T + I + F \leq 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 1.9 (n-SuperHyperNeutrosophic Set)**.** *Scenario:* Multi-level hierarchical analysis of climate change impacts.

Example: Let $X = \{\text{Temperature, Rainfall, Sea Level}\}$, representing key factors influenced by climate change. We consider a three-level hierarchy:

- *Level 1:* Regions {Region 1, Region 2}.
- *Level 2:* Countries within regions, e.g., {Country A, Country B, Country C}.
- *Level 3:* Cities within countries, e.g., {City X, City Y, City Z}.

For each level, the n-SuperHyperNeutrosophic Set assigns a family of subsets with membership triplets. For instance:

$$\tilde{A}_3(\text{City X}) = \{(0.8, 0.15, 0.05), (0.7, 0.2, 0.1)\},$$

where each triplet represents truth (T), indeterminacy (I), and falsity (F) degrees at the city level. This approach integrates uncertainty at regional, country, and city scales for holistic decision-making.

2 Results of This Paper

This section outlines the main results presented in this paper.

2.1 Nonstationary Neutrosophic Set

A Nonstationary Neutrosophic Set extends the neutrosophic set by introducing time-dependent truth, indeterminacy, and falsity memberships, which dynamically evolve over a time domain T. Related concepts, such as Nonstationary Fuzzy Sets [4, 4, 51, 57, 58, 95], are also well-known in the literature.

Definition 2.1 (Nonstationary Neutrosophic Set)**.** [37] Let X be a non-empty set, and let T be a time domain (which may be continuous or discrete). A *nonstationary neutrosophic set* $\dot{A}$ on X is defined by three *time-dependent* membership functions:

$$T_{\dot{A}} : T \times X \to [0,1], \quad I_{\dot{A}} : T \times X \to [0,1], \quad F_{\dot{A}} : T \times X \to [0,1],$$

where for each $(t,x) \in T \times X$, the values $T_{\dot{A}}(t,x)$, $I_{\dot{A}}(t,x)$, and $F_{\dot{A}}(t,x)$ represent the *truth*, *indeterminacy*, and *falsity* degrees of x in $\dot{A}$ at time t. These satisfy

$$0 \le T_{\dot{A}}(t,x) + I_{\dot{A}}(t,x) + F_{\dot{A}}(t,x) \le 3,$$

for all $(t,x) \in T \times X$.

Analogous to the nonstationary fuzzy set, each component can be viewed as a *time-varying perturbation* of the corresponding membership function in a *stationary* neutrosophic set. Specifically, if T_A, I_A, and F_A define a classical neutrosophic set A on X (with no time dependence), then we introduce a set of time-dependent parameters

$$\{p_{T,i}(t),\ p_{I,j}(t),\ p_{F,k}(t)\} \quad \text{for } i,j,k \in \mathcal{I},$$

and define

$$\begin{aligned} T_{\dot{A}}(t,x) &= T_A\big(x;\, p_{T,1}(t), \ldots, p_{T,m}(t)\big), \\ I_{\dot{A}}(t,x) &= I_A\big(x;\, p_{I,1}(t), \ldots, p_{I,n}(t)\big), \\ F_{\dot{A}}(t,x) &= F_A\big(x;\, p_{F,1}(t), \ldots, p_{F,p}(t)\big), \end{aligned}$$

where each parameter function $p_{\cdot,\cdot}(t)$ may evolve over time via a perturbation rule, e.g.

$$p_{T,i}(t) = p_{T,i} + k_{T,i} \cdot f_{T,i}(t),$$

and similarly for the indeterminacy and falsity parameters. In integral notation, the nonstationary neutrosophic set $\dot{A}$ can be expressed as

$$\dot{A} = \int_{t\in T} \int_{x\in X} \big(T_{\dot{A}}(t,x),\, I_{\dot{A}}(t,x),\, F_{\dot{A}}(t,x)\big) / x \,/\, t.$$

Definition 2.2 (Nonstationary HyperNeutrosophic Set)**.** Let X be a non-empty set, and let T be a time domain (continuous or discrete). A *Nonstationary HyperNeutrosophic Set* $\dot{\tilde{A}}$ on X is a mapping

$$\dot{\tilde{\mu}} : T \times X \longrightarrow \mathcal{P}\big([0,1]^3\big),$$

where for each pair $(t,x) \in T \times X$,

$$\dot{\tilde{\mu}}(t,x) \subseteq [0,1]^3,$$

and every element $\big(T_{t,x}, I_{t,x}, F_{t,x}\big) \in \dot{\tilde{\mu}}(t,x)$ satisfies

$$0 \le T_{t,x} + I_{t,x} + F_{t,x} \le 3.$$

Furthermore, each $\dot{\tilde{\mu}}(t,x)$ can *evolve in time*, meaning that for different values of t, the subsets $\dot{\tilde{\mu}}(t,x) \subseteq [0,1]^3$ may vary.

Interpretation.

- This definition *generalizes* the concept of a Nonstationary Neutrosophic Set by allowing each membership evaluation at time t and element x to be not just a single triple (T, I, F), but a *set* of possible membership triples in $[0,1]^3$.
- It *simultaneously* generalizes the concept of a HyperNeutrosophic Set by introducing a time dimension and making the hyperneutrosophic membership $\dot{\tilde{\mu}}(t,x)$ *time-dependent*.

Theorem 2.3 (Nonstationary HyperNeutrosophic Set Generalizes Both Nonstationary Neutrosophic Set and HyperNeutrosophic Set)**.**

1. *If, for each* (t, x)*, the set* $\dot{\tilde{\mu}}(t, x) \subseteq [0, 1]^3$ *is always a singleton, i.e.*

$$\dot{\tilde{\mu}}(t, x) = \{(T_{\dot{A}}(t, x),\ I_{\dot{A}}(t, x),\ F_{\dot{A}}(t, x))\},$$

   then we recover a Nonstationary Neutrosophic Set*.*

2. *If we* freeze time *to a single value* $t = t_0$*, then* $\dot{\tilde{\mu}}(t_0, x) \subseteq [0, 1]^3$ *describes a standard HyperNeutrosophic Set (with no time dependence).*

Proof. (1) Suppose each $\dot{\tilde{\mu}}(t, x)$ is a singleton set. Then for each $(t, x) \in T \times X$, there is exactly one triple $(T_{t,x}, I_{t,x}, F_{t,x})$. This precisely recovers the definition of a Nonstationary Neutrosophic Set.

(2) Conversely, if we consider a time domain consisting of a single time instant t_0, then $\dot{\tilde{\mu}}(t_0, x) \subseteq [0, 1]^3$ for each x. This yields exactly the definition of a HyperNeutrosophic Set $\tilde{A}$ with membership $\tilde{A}(x) = \dot{\tilde{\mu}}(t_0, x)$. Hence the Nonstationary HyperNeutrosophic structure generalizes both concepts. □

Definition 2.4 (Nonstationary n-SuperHyperNeutrosophic Set)**.** Let X be a non-empty set and let T be a time domain (continuous or discrete). A *Nonstationary n-SuperHyperNeutrosophic Set* $\dot{\tilde{A}}_n$ is defined as a mapping

$$\dot{\tilde{A}}_n : T \times \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n([0, 1]^3),$$

subject to:

- $\mathcal{P}_n(X)$ is the n-th powerset of X, recursively constructed via:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_{k+1}(X) = \mathcal{P}\Big(\mathcal{P}_k(X)\Big), \quad k \geq 1.$$

- $\mathcal{P}_n([0, 1]^3)$ is the n-th powerset of the unit cube $[0, 1]^3$.
- For each $(t, A) \in T \times \mathcal{P}_n(X)$,

$$\dot{\tilde{A}}_n(t, A) \subseteq \mathcal{P}_n([0, 1]^3).$$

- All neutrosophic membership triples $(T_{(t,A)},\ I_{(t,A)},\ F_{(t,A)}) \in [0, 1]^3$ must satisfy

$$0 \leq T_{(t,A)} + I_{(t,A)} + F_{(t,A)} \leq 3.$$

Moreover, each membership assignment $\dot{\tilde{A}}_n(t, A) \subseteq \mathcal{P}_n([0, 1]^3)$ can vary with time $t \in T$, making the set *nonstationary*.

Theorem 2.5 (Nonstationary n-SuperHyperNeutrosophic Set Generalizes Both Nonstationary HyperNeutrosophic Set and n-SuperHyperNeutrosophic Set)**.**

1. *By restricting* $n = 1$ *in Definition 2.4, we obtain a* Nonstationary HyperNeutrosophic Set*.*
2. *By restricting the time domain T to a single value* t_0*, we obtain a standard* $(n-SuperHyperNeutrosophicSet)$ *with no time dependence.*

Proof. (1) If $n = 1$*,* then $\mathcal{P}_1(X) = \mathcal{P}(X)$ and $\mathcal{P}_1([0, 1]^3) = \mathcal{P}([0, 1]^3)$. Thus the mapping

$$\dot{\tilde{A}}_1 : T \times \mathcal{P}(X) \longrightarrow \mathcal{P}([0, 1]^3)$$

coincides with the definition of a *Nonstationary HyperNeutrosophic Set* in Definition 2.2.

(2) If we fix a single time instant $t_0 \in T$, then for each $A \in \mathcal{P}_n(X)$, we assign

$$\dot{\tilde{A}}_n(t_0, A) \subseteq \mathcal{P}_n([0, 1]^3).$$

Hence the membership is no longer time-dependent, becoming

$$\tilde{A}_n(A) = \dot{\tilde{A}}_n(t_0, A).$$

This is precisely the definition of an $(n - SuperHyperNeutrosophicSet)$ that maps $\mathcal{P}_n(X)$ into $\mathcal{P}_n([0, 1]^3$. Therefore, the Nonstationary dimension is removed, and we recover the stationary case. □

Theorem 2.6 (Time-Frozen or Hyper-Frozen Slices)**.** *Let $\dot{\tilde{A}}_n$ be a Nonstationary n-SuperHyperNeutrosophic Set.*

1. Time-Frozen Slices: *For each fixed $t \in T$, the structure $\dot{\tilde{A}}_n(t, \cdot)$ yields an ordinary $(n{-}SuperHyperNeutrosophicSet)$ on $\mathcal{P}_n(X)$.*
2. Hyper-Frozen Slices: *For each fixed sub-collection $C \subseteq \mathcal{P}_n(X)$ restricted to singletons or first-level sets, we recover a Nonstationary (but not superhyper) membership arrangement, akin to a Nonstationary Neutrosophic Set or Nonstationary HyperNeutrosophic Set.*

Proof. (1) Follows directly from Theorem 2.5 part (2). *(2)* If we restrict the hyper operation such that only single-level or single-subset outputs are allowed, the superhyper approach collapses to a simpler hyper or non-hyper structure, preserving the time dependence. The details align with the usual embeddings from superhyper to hyper or classical sets. □

2.2 Type-m Neutrosophic Set

A Type-m Neutrosophic Set hierarchically represents truth, indeterminacy, and falsity memberships by recursively nesting Type-$(m-1)$ Neutrosophic Sets for $m \geq 1$ [44]. Related concepts include the Type-2 Neutrosophic Set [1, 18, 55, 63], Type-2 Fuzzy Set [12, 19–21, 53, 64, 66, 66, 69, 70, 72, 92–94], Type-3 Fuzzy Set [14–17, 71, 74], and Type-m Fuzzy Set [2, 75–77, 91, 105]. The definition is provided below.

Definition 2.7 (Type-m Neutrosophic Set)**.** [44] Let X be a non-empty universe of discourse, and let $m \geq 1$. A *Type-m Neutrosophic Set (TMS)* $\mathcal{A}$ on X is defined by three membership functions

$$T_{\mathcal{A}},\ I_{\mathcal{A}},\ F_{\mathcal{A}} : X \longrightarrow \mathcal{M}_{m-1}[0,1],$$

where $\mathcal{M}_{m-1}[0,1]$ denotes the set of all *Type-$(m-1)$ Neutrosophic Sets* whose truth, indeterminacy, and falsity memberships lie in $[0,1]$. Formally,

$$\mathcal{A} \;=\; \bigl\{\, \langle x,\ T_{\mathcal{A}}(x),\ I_{\mathcal{A}}(x),\ F_{\mathcal{A}}(x)\rangle \;\mid\; x \in X \bigr\}.$$

Moreover, at any *terminal level* (i.e. when $m = 1$), the membership functions become

$$T_{\mathcal{A}}(x),\ I_{\mathcal{A}}(x),\ F_{\mathcal{A}}(x) \;\in\; [0,1],$$

subject to the constraint

$$0 \;\leq\; T_{\mathcal{A}}(x) \;+\; I_{\mathcal{A}}(x) \;+\; F_{\mathcal{A}}(x) \;\leq\; 3.$$

This recursion implies that each $(T_{\mathcal{A}}, I_{\mathcal{A}}, F_{\mathcal{A}})$ at the $(m-1)$-level is itself a Type-$(m-1)$ Neutrosophic Set, until the recursion terminates at $m = 1$.

2.3 Subset-Valued Neutrosophic Set

Subset-Valued Neutrosophic Set (SSVNS) represents truth, indeterminacy, and falsity as subsets of $[0,1]$, satisfying bounded sum constraints [30].

Definition 2.8 (Subset-Valued Neutrosophic Set)**.** [86, 88] Let X be a space of elements, where $x \in X$ represents a generic element. A Subset-Valued Neutrosophic Set (SSVNS) A is characterized by the following membership functions:

- The *truth membership function* $T_A(x)$,
- The *indeterminacy membership function* $I_A(x)$,
- The *falsity membership function* $F_A(x)$,

where each function satisfies the conditions:

$$T_A(x), I_A(x), F_A(x) \subseteq [0,1], \quad \forall x \in X,$$

and

$$0 \leq \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \leq 3.$$

Definition 2.9 (Subset-Valued HyperNeutrosophic Set (SV-HNS))**.** Let X be a non-empty set, and denote $\mathcal{P}_{\neq\emptyset}\big([0,1]\big)$ as the set of all non-empty subsets of $[0,1]$. We define

$$\mathcal{P}^3_{\neq\emptyset}([0,1]) \;=\; \Big\{ \big(T, I, F\big) \;\big|\; T, I, F \in \mathcal{P}_{\neq\emptyset}\big([0,1]\big) \Big\},$$

the space of *non-empty subset-triples*. A *Subset-Valued HyperNeutrosophic Set (SV-HNS)* $\widetilde{A}$ is a mapping

$$\widetilde{\mu} : X \;\longrightarrow\; \mathcal{P}\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big) \setminus \{\varnothing\},$$

where for each $x \in X$,

$$\widetilde{\mu}(x) \;\subseteq\; \Big\{ (T, I, F) \;\Big|\; T \cup I \cup F \subseteq [0,1],\ \sup(T) + \sup(I) + \sup(F) \;\leq\; 3 \Big\},$$

and $\widetilde{\mu}(x) \neq \varnothing$. Equivalently, each $\widetilde{\mu}(x)$ is a *set of subset-triples* $\big(T, I, F\big)$, each triple satisfying

- $T, I, F \subseteq [0,1]$ (non-empty),
- $\sup(T) + \sup(I) + \sup(F) \leq 3$.

Remark 2.10.

- This definition *generalizes* the standard hyperneutrosophic notion by replacing each numerical triple $(T, I, F) \in [0,1]^3$ with subset-triples $\big(T, I, F\big)$ with each of $T, I, F \subseteq [0,1]$.
- The output of $\widetilde{\mu}(x)$ is not a single subset-triple; it is a *set* of such subset-triples. This is the essence of the hyper notion: one element in X can be mapped to multiple membership descriptions simultaneously.

Theorem 2.11 (Subset-Valued HyperNeutrosophic Set generalizes Subset-Valued Neutrosophic Set)**.** *If for each $x \in X$, the hyperneutrosophic mapping $\widetilde{\mu}(x)$ contains exactly one subset-triple $\big(T_A(x), I_A(x), F_A(x)\big)$, then we recover the standard* Subset-Valued Neutrosophic Set *from Definition.*

Proof. In Subset-Valued HyperNeutrosophic structure, we allow $\widetilde{\mu}(x) \subseteq \mathcal{P}^3_{\neq\emptyset}([0,1])$. If we force $\widetilde{\mu}(x)$ to be a singleton for all x, i.e.

$$\widetilde{\mu}(x) = \big\{\, (T_A(x), I_A(x), F_A(x)) \big\},$$

then there is exactly one triple of subsets describing the membership for each x. This precisely recovers the Subset-Valued Neutrosophic Set, where each x is assigned a unique triple $\big(T_A(x), I_A(x), F_A(x)\big)$. Hence, restricting the hyper notion to singletons collapses the mapping to a single triple per x, i.e., the standard subset-valued membership approach. □

Definition 2.12 (Subset-Valued n-SuperHyperNeutrosophic Set (SV-nSHNS))**.** Let X be a non-empty set, and let $\mathcal{P}_1(X) = \mathcal{P}(X)$, $\mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big)$ for $k \geq 1$. Also define a similar iterative process for subset-triples in $[0,1]$. A *Subset-Valued n-SuperHyperNeutrosophic Set (SV-nSHNS)* is a map

$$\widetilde{A}_n : \; \mathcal{P}_n(X) \;\;\longrightarrow\;\; \mathcal{P}_n\Big(\mathcal{P}^3_{\neq\emptyset}\big([0,1]\big)\Big),$$

subject to:

- For each $A \in \mathcal{P}_n(X)$, $\widetilde{A}_n(A) \subseteq \mathcal{P}_n\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big)$,
- Each *element* of $\widetilde{A}_n(A)$ (call it a *hyper-element*) must be a set of subset-triples $\big(T, I, F\big)$ that satisfy

$$T, I, F \subseteq [0,1], \quad T \cup I \cup F \subseteq [0,1], \quad \sup(T) + \sup(I) + \sup(F) \;\leq\; 3.$$

Equivalently, each $\widetilde{A}_n(A)$ is a *(possibly large) set* whose elements are themselves *collections* of subset-based membership triplets, arranged in up to n-level nested hyper-structures.

Remark 2.13.
- This structure is both *subset-valued* (since each membership is a triple of subsets of $[0,1]$) and *n-superhyper* (since we are mapping from $\mathcal{P}_n(X)$ to an n-level powerset of subset-triples).
- It significantly generalizes the standard n-SuperHyperNeutrosophic Set from $\mathcal{P}_n(X)$ to $\mathcal{P}_n([0,1]^3)$.

Theorem 2.14 (Subset-Valued n-SuperHyperNeutrosophic Set generalizes Subset-Valued HyperNeutrosophic Set)**.** *Let $\widetilde{A}_n$ be a Subset-Valued n-SuperHyperNeutrosophic Set (Definition 2.12). If $n = 1$, we recover the* Subset-Valued HyperNeutrosophic Set (SV-HNS) *from Definition 2.9.*

Proof. If $n = 1$, then $\mathcal{P}_1(X) = \mathcal{P}(X)$, and $\mathcal{P}_1\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big) = \mathcal{P}\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big)$. Thus the mapping

$$\widetilde{A}_1 : \mathcal{P}(X) \longrightarrow \mathcal{P}\Big(\mathcal{P}^3_{\neq\emptyset}([0,1])\Big) \setminus \{\emptyset\}.$$

But $\mathcal{P}(X)$ in the domain is effectively the set X if we interpret each singleton $\{x\} \in \mathcal{P}(X)$ as an element. By identifying each $A \in \mathcal{P}(X)$ with a single point $x \in X$, we replicate the Subset-Valued *HyperNeutrosophic* approach. More rigorously, the standard hyperneutrosophic definition from Definition 2.9 uses a map

$$\widetilde{\mu} : X \to \mathcal{P}\big(\mathcal{P}^3_{\neq\emptyset}([0,1])\big),$$

where each $\widetilde{\mu}(x) \subseteq \mathcal{P}^3_{\neq\emptyset}([0,1])$. Setting $n = 1$ identifies each element $x \in X$ with the subset $\{x\} \in \mathcal{P}(X)$. Hence the domain collapses to first-level subsets, and we precisely recover the Subset-Valued HyperNeutrosophic Set structure. □

Theorem 2.15 (Subset-Valued HyperNeutrosophic Set generalizes Subset-Valued Neutrosophic Set)**.** *If each mapping $\widetilde{\mu}(x) \subseteq \mathcal{P}^3_{\neq\emptyset}([0,1])$ is restricted to exactly one triple of subsets, we recover the standard Subset-Valued Neutrosophic Set.*

Proof. This was essentially Theorem 2.11, restated. When the hyper mapping is forced to be a singleton for each x, we get a single subset-triple $\big(T_A(x), I_A(x), F_A(x)\big)$. That exactly matches Definition. □

Theorem 2.16 (Subset-Valued n-SuperHyperNeutrosophic Set extends standard n-SuperHyperNeutrosophic Set)**.** *By imposing that each membership triple $\big(T, I, F\big) \in [0,1]^3$ be single-valued subsets (i.e., each is a single-point subset), a Subset-Valued n-SuperHyperNeutrosophic Set reduces to an n-SuperHyperNeutrosophic Set as defined in prior works.*

Proof. In a Subset-Valued n-SuperHyperNeutrosophic Set $\widetilde{A}_n$, each membership hyper-value is an element in

$$\mathcal{P}_n\big(\mathcal{P}^*([0,1]^3)\big),$$

where $\mathcal{P}^*([0,1]^3)$ denotes the power set of $[0,1]^3$ excluding the empty set. If we impose the condition that each subset $\{T, I, F\} \subseteq [0,1]$ is a singleton (i.e., for each coordinate T, I, F, the subset contains exactly one element), then any membership triple (T, I, F) effectively becomes $\{(\tau, \iota, \varphi)\} \subseteq [0,1]^3$, where (τ, ι, φ) is a single point in the unit cube.

This restriction collapses the subset-based membership values to single-point numerical values. Consequently, the mapping domain $\mathcal{P}_n(X)$ to $\mathcal{P}_n([0,1]^3)$ of the Subset-Valued n-SuperHyperNeutrosophic Set coincides with the mapping used in a standard n-SuperHyperNeutrosophic Set. Thus, the subset-based approach is a generalization of the numerical-based approach. □

2.4 Single-Valued Complex Refined Neutrosophic Set

Single-Valued Complex Refined Neutrosophic Set (SVCRNS) refines truth, indeterminacy, and falsity into sub-functions with real (amplitude) and imaginary (phase) components [86]

Definition 2.17 (Single-Valued Complex Refined Neutrosophic Set (SVCRNS))**.** [86] Let X be a space of elements, with $x \in X$ representing a generic element. A *Single-Valued Complex Refined Neutrosophic Set (SVCRNS) A* is defined as follows:

Characteristics

1. *p-Sub-truth Membership Functions:* Each truth membership function is expressed in complex form:

$$T_{A1}(x)e^{iT^I_{A1}(x)}, T_{A2}(x)e^{iT^I_{A2}(x)}, \ldots, T_{Ap}(x)e^{iT^I_{Ap}(x)},$$

   where $T_{Ai}(x) \in [0,1]$ represents the real part (amplitude), and $T^I_{Ai}(x) \in [0, 2\pi]$ represents the imaginary part (phase).

2. *r-Sub-indeterminacy Membership Functions:* Each indeterminacy membership function is expressed in complex form:

$$I_{A1}(x)e^{iI^I_{A1}(x)}, I_{A2}(x)e^{iI^I_{A2}(x)}, \ldots, I_{Ar}(x)e^{iI^I_{Ar}(x)},$$

   where $I_{Aj}(x) \in [0,1]$ and $I^I_{Aj}(x) \in [0, 2\pi]$.

3. *s-Sub-falsity Membership Functions:* Each falsity membership function is expressed in complex form:

$$F_{A1}(x)e^{iF^I_{A1}(x)}, F_{A2}(x)e^{iF^I_{A2}(x)}, \ldots, F_{As}(x)e^{iF^I_{As}(x)},$$

   where $F_{Ak}(x) \in [0,1]$ and $F^I_{Ak}(x) \in [0, 2\pi]$.

Conditions

1. *Real Part Condition (Amplitude):* For all $x \in X$, the real parts satisfy:

$$0 \le \sum_{i=1}^{p} T_{Ai}(x) + \sum_{j=1}^{r} I_{Aj}(x) + \sum_{k=1}^{s} F_{Ak}(x) \le p + r + s.$$

2. *Imaginary Part Condition (Phase):* For all $x \in X$, the imaginary parts satisfy:

$$0 \le \sum_{i=1}^{p} T^I_{Ai}(x) + \sum_{j=1}^{r} I^I_{Aj}(x) + \sum_{k=1}^{s} F^I_{Ak}(x) \le 2\pi(p + r + s).$$

3. *Parameter Constraints:* The integers $p, r, s \ge 0$, and at least one of p, r, s satisfies ≥ 2:

$$\max(p, r, s) \ge 2.$$

Explanation

- The real part (or amplitude) $T_{Ai}(x), I_{Aj}(x), F_{Ak}(x)$ represents the intensity or degree of truth, indeterminacy, and falsity for each sub-membership function.
- The imaginary part (or phase) $T^I_{Ai}(x), I^I_{Aj}(x), F^I_{Ak}(x)$ introduces a cyclic or periodic component, extending the membership functions into the complex domain.
- The refinement into $p+r+s$ sub-functions allows for a more detailed and multidimensional representation of truth, indeterminacy, and falsity.

Definition 2.18 (Single-Valued Complex Refined HyperNeutrosophic Set (SVCR-HNS))**.** Let X be a non-empty set. Denote by $C_{p,r,s}$ the set of all possible refined complex membership tuples

$$\Big(T_1 e^{i\,T^I_1}, \ldots, T_p e^{i\,T^I_p} \;\big|\; I_1 e^{i\,I^I_1}, \ldots, I_r e^{i\,I^I_r} \;\big|\; F_1 e^{i\,F^I_1}, \ldots, F_s e^{i\,F^I_s}\Big)$$

satisfying the amplitude-phase constraints in Definition. A *Single-Valued Complex Refined HyperNeutrosophic Set (SVCR-HNS)* $\widetilde{H}$ is a mapping

$$\widetilde{H} : X \longrightarrow \mathcal{P}\big(C_{p,r,s}\big) \setminus \{\emptyset\},$$

where for each $x \in X$,

$$\widetilde{H}(x) \subseteq \Big\{\big(T_{A1}(x)e^{iT^I_{A1}(x)}, \ldots, F_{As}(x)e^{iF^I_{As}(x)}\big) : \text{satisfying refined constraints}\Big\},$$

and $\widetilde{H}(x) \neq \emptyset$. In other words, $\widetilde{H}(x)$ is a (possibly large) set of complex refined membership tuples, each composed of p-sub-truth values, r-sub-indeterminacy values, and s-sub-falsity values in the complex plane, subject to amplitude ≤ 1 and phase $\le 2\pi$.

Theorem 2.19 (SVCR-HNS generalizes SVCRNS)**.** *If for each $x \in X$, $\widetilde{H}(x)$ is restricted to exactly one refined complex membership tuple $\{ (T_{A1}(x)e^{iT^I_{A1}(x)}, \dots) \}$, then the SVCR-HNS reduces to a Single-Valued Complex Refined Neutrosophic Set from Definition.*

Proof. By definition, an SVCR-HNS $\widetilde{H}(x)$ is a non-empty set of possible refined membership tuples. If we force $\widetilde{H}(x)$ to be a singleton for each $x \in X$, then each element has exactly one (p, r, s)-refined membership tuple, which is precisely the single-valued mapping in the original sense. The amplitude-phase constraints remain the same, so we get back to a single, refined complex membership assignment. □

Definition 2.20 (Single-Valued Complex Refined n-SuperHyperNeutrosophic Set (SVCR-nSHNS))**.** Let X be a non-empty set, and define:

$$\mathcal{P}_1(X) = \mathcal{P}(X), \quad \mathcal{P}_{k+1}(X) = \mathcal{P}\big(\mathcal{P}_k(X)\big) \quad \text{for } k \geq 1.$$

Similarly, define a space of refined complex membership structures:

$$C_{p,r,s},$$

and iteratively:

$$\mathcal{P}_n\big(C_{p,r,s}\big) = \underbrace{\mathcal{P}\big(\cdots\mathcal{P}\big)}_{n \text{ times}}(C_{p,r,s})\big)\cdots\big).$$

A *Single-Valued Complex Refined* ***n****-SuperHyperNeutrosophic Set (SVCR-****n****SHNS)* is defined as a mapping:

$$\widetilde{H}_n : \mathcal{P}_n(X) \longrightarrow \mathcal{P}_n\big(C_{p,r,s}\big) \setminus \{\emptyset\},$$

where each image $\widetilde{H}_n(A)$ (for $A \in \mathcal{P}_n(X)$) is a non-empty subset of $C_{p,r,s}$ arranged in up to n-level nested hyper-structures. Each membership tuple within $\widetilde{H}_n(A)$ must satisfy the following conditions:

$$\begin{aligned}
&T_{Ai}(A), I_{Aj}(A), F_{Ak}(A) \in [0, 1],\\
&T^I_{Ai}(A), I^I_{Aj}(A), F^I_{Ak}(A) \in [0, 2\pi],\\
&0 \;\leq\; \sum_{i=1}^{p} T_{Ai}(A) + \sum_{j=1}^{r} I_{Aj}(A) + \sum_{k=1}^{s} F_{Ak}(A) \;\leq\; p + r + s,\\
&0 \;\leq\; \sum_{i=1}^{p} T^I_{Ai}(A) + \sum_{j=1}^{r} I^I_{Aj}(A) + \sum_{k=1}^{s} F^I_{Ak}(A) \;\leq\; 2\pi\,(p + r + s).
\end{aligned}$$

Hence, the domain is $\mathcal{P}_n(X)$, and the codomain is $\mathcal{P}_n\big(C_{p,r,s}\big)$.

Theorem 2.21 (SVCR-nSHNS generalizes SVCR-HNS)**.** *If $n = 1$ in Definition 2.20, we recover the Single-Valued Complex Refined HyperNeutrosophic Set (SVCR-HNS) from Definition 2.18.*

Proof. By setting $n = 1$, we have $\mathcal{P}_1(X) = \mathcal{P}(X)$ and $\mathcal{P}_1\big(C_{p,r,s}\big) = \mathcal{P}\big(C_{p,r,s}\big)$. Thus the map

$$\widetilde{H}_1 : \mathcal{P}(X) \longrightarrow \mathcal{P}\big(C_{p,r,s}\big) \setminus \{\emptyset\}$$

becomes precisely the SVCR-HNS definition, except each element of $\mathcal{P}(X)$ is identified with a single point $x \in X$. Indeed, a standard identification is that each $x \in X$ is in one-to-one correspondence with $\{x\} \in \mathcal{P}(X)$. Thus the domain effectively reduces to X in the sense of hyperneutrosophic structures, and we get the SVCR-HNS. □

Theorem 2.22 (SVCR-HNS generalizes SVCRNS)**.** *If each $\widetilde{H}(x)$ in an SVCR-HNS is restricted to a singleton refined complex membership tuple, then we recover the Single-Valued Complex Refined Neutrosophic Set (SVCRNS) from Definition.*

Proof. This was established in Theorem 2.19 but restated here for completeness. Imposing a single membership tuple per element collapses the hyper-based approach to the single-valued approach. □

Theorem 2.23 (SVCR-nSHNS generalizes all simpler forms)**.** *An SVCR-nSuperHyperNeutrosophic Set can be specialized to any of:*

1. *An SVCR-HNS (by setting $n = 1$),*
2. *An SVCRNS (by setting $n = 1$ and forcing singletons in the hyper sense),*
3. *A standard n-SuperHyperNeutrosophic set (by ignoring the complex and refined sub-components, i.e. setting $p = r = s = 1$ with zero phase).*

Proof. *(1)* $\boldsymbol{n = 1}$ *yields SVCR-HNS:* Directly from Definition 2.20, $\mathcal{P}_1(X) = \mathcal{P}(X)$ and the codomain is $\mathcal{P}_1(C_{p,r,s})$. This matches the hyper definition for single-layer. *(2) Singleton restriction yields SVCRNS:* If each set in the codomain has exactly one refined complex membership tuple, then the hyper notion collapses, leaving a single complex refined membership triple per element (Theorem 2.22). *(3) Setting* $\boldsymbol{p = r = s = 1}$ *and phases to zero yields the standard real-valued superhyperneutrosophic approach:* If each amplitude $= T, I, F \in [0, 1]$ and each phase $\equiv 0$, then each sub-truth membership is just a single real number in $[0, 1]$. Summation constraints revert to the standard T+I+F $\leq$ 3. The superhyper aspect remains in domain and codomain recursion, but the refined complex perspective is gone. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Mohamed Abdel-Basset, M Saleh, Abduallah Gamal, and Florentin Smarandache. An approach of topsis technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing*, 77:438–452, 2019.

[2] Janet Aisbett, John T Rickard, and David Morgenthaler. Intersection and union of type-n fuzzy sets. In *International Conference on Fuzzy Systems*, pages 1–8. IEEE, 2010.

[3] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[4] Marcos Antônio Alves, Petrônio Cândido de Lima e Silva, Carlos Alberto Severiano Junior, Gustavo Linhares Vieira, Frederico Gadelha Guimarães, and Hossein Javedani Sadaei. An extension of nonstationary fuzzy sets to heteroskedastic fuzzy time series. In *ESANN*, 2018.

[5] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[6] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[7] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[8] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[9] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[10] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[11] Vivek Badhe, RS Thakur, and GS Thakur. Vague set theory for profit pattern and decision making in uncertain data. *International journal of advanced computer science and applications*, 6(6):58–64, 2015.

[12] Adil Baykasolu and Ilker Golcuk. Development of an interval type-2 fuzzy sets based hierarchical madm model by combining dematel and topsis. *Expert Syst. Appl.*, 70:37–51, 2017.

[13] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[14] Oscar Castillo, Juan R Castro, and Patricia Melin. Interval type-3 fuzzy control for automated tuning of image quality in televisions. *Axioms*, 11(6):276, 2022.

[15] Oscar Castillo and Patricia Melin. Interval type-3 fuzzy decision-making in material surface quality control. In *Virtual International Conference on Soft Computing, Optimization Theory and Applications*, pages 157–169. Springer, 2021.

[16] Oscar Castillo and Patricia Melin. Type-3 fuzzy theory. In *Type-3 Fuzzy Logic in Intelligent Control*, pages 5–15. Springer, 2023.

[17] Oscar Castillo and Patricia Melin. Type-3 for prediction. In *Type-3 Fuzzy Logic in Time Series Prediction*, pages 5–15. Springer, 2024.

[18] Soumen Kumar Das, F Yu Vincent, Sankar Kumar Roy, and Gerhard Wilhelm Weber. Location–allocation problem for green efficient two-stage vehicle-based logistics system: A type-2 neutrosophic multi-objective modeling approach. *Expert Systems with Applications*, 238:122174, 2024.

[19] Arnab Kumar De, Debjani Chakraborty, and Animesh Biswas. Literature review on type-2 fuzzy set theory. *Soft Computing*, 26:9049 – 9068, 2022.

[20] Muhammet Deveci, Umit Cali, Sadik Kucuksari, and Nuh Erdogan. Interval type-2 fuzzy sets based multi-criteria decision-making model for offshore wind farm development in ireland. *Energy*, 198:117317, 2020.

[21] Juan Carlos FigueroaGarcía, Jennifer Soraya Ramos-Cuesta, and G HernándezPérez. On the variance of interval type-2 fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 31:2320–2330, 2023.

[22] Ronald C. Freiwald. An introduction to set theory and topology. 2014.

[23] Takaaki Fujita. A concise review on various concepts of superhyperstructures.

[24] Takaaki Fujita. Some types of hyperfuzzy set (2): Nonstationary, dual hesitant, interval-valued, and l-fuzzy set. *Preprint*.

[25] Takaaki Fujita. Some types of hyperfuzzy set: Bipolar, m-polar, q-rung orthopair, trapezoidal, linguistic, intuitionistic, picture, hesitant, spherical, type-m, offset, overset, and underset. *Preprint*.

[26] Takaaki Fujita. Some types of hyperneutrosophic set (4): Cubic, trapezoidal, q-rung orthopair, overset, underset, and offset.

[27] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[28] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[29] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[30] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[31] Takaaki Fujita. Short note of bunch graph in fuzzy, neutrosophic, and plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2024.

[32] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[33] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[34] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. 2025.

[35] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[36] Takaaki Fujita. Hyperfuzzy hyperrough set, hyperneutrosophic hyperrough set, and hypersoft hyperrough set. Preprint, 2025.

[37] Takaaki Fujita. L-neutrosophic set and nonstationary neutrosophic set. 2025.

[38] Takaaki Fujita. Short note of even-hole-graph for uncertain graph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 351, 2025.

[39] Takaaki Fujita. Some type of hyperneutrosophic set: Bipolar, pythagorean, double-valued, interval-valued set, 2025. Preprint.

[40] Takaaki Fujita. Some types of hyperneutrosophic set (2): Complex, single-valued triangular, fermatean, and linguistic sets. *Preprint*, 2025.

[41] Takaaki Fujita. Some types of hyperneutrosophic set (3): Dynamic, quadripartitioned, pentapartitioned, heptapartitioned, m-polar. 2025.

[42] Takaaki Fujita. Some types of hyperneutrosophic set (5): Support, paraconsistent, faillibilist, and others, 2025. Preprint.

[43] Takaaki Fujita. Some types of hyperneutrosophic set (6): Multineutrosophic set and refined neutrosophic set, 2025. Preprint.

[44] Takaaki Fujita. Type-n fuzzy offset, neutrosophic offset, and plithogenic offset. 2025.

[45] Takaaki Fujita and Florentin Smarandache. A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 308.

[46] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[47] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*, 2024.

[48] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[49] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[50] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[51] Jonathan M Garibaldi, Marcin Jaroszewski, and Salang Musikasuwan. Nonstationary fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 16(4):1072–1086, 2008.

[52] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[53] Mehdi Keshavarz Ghorabaee, Maghsoud Amiri, Edmundas Kazimieras Zavadskas, Zenonas Turskis, and Jurgita Antuchevicien. A new multi-criteria model based on interval type-2 fuzzy sets and edas method for supplier evaluation and order allocation with environmental considerations. *Comput. Ind. Eng.*, 112:156–174, 2017.

[54] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[55] Marwan H Hassan, Saad M Darwish, and Saleh M Elkaffas. Type-2 neutrosophic set and their applications in medical databases deadlock resolution. *Computers, Materials & Continua*, 74(2), 2023.

[56] Karel Hrbacek and Thomas Jech. Introduction to set theory, revised and expanded. 2017.

[57] Han-Chen Huang and Xiao-Jun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, 24:213–228, 2016.

[58] Han-Chen Huang and Xiaojun Yang. A comparative investigation of type-2 fuzzy sets, nonstationary fuzzy sets and cloud models. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 24(02):213–227, 2016.

[59] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[60] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[61] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Length-fuzzy subalgebras in bck/bci-algebras. *Mathematics*, 6(1):11, 2018.

[62] Jeong Gi Kang and Hashem Bordbar. Further results on length fuzzy subalgebras of bck/bci-algebras. 2018.

[63] Faruk Karaaslan and Fatih Hunu. Type-2 single-valued neutrosophic sets and their applications in multi-criteria group decision making based on topsis method. *Journal of Ambient Intelligence and Humanized Computing*, 11:4113–4132, 2020.

[64] Nilesh N. Karnik and Jerry M. Mendel. Operations on type-2 fuzzy sets. *Fuzzy Sets Syst.*, 122:327–348, 2001.

[65] Kazimierz Kuratowski. Introduction to set theory and topology. 1964.

[66] Hao Li, Xianchao Dai, Ligang Zhou, and Qun Wu. Encoding words into interval type-2 fuzzy sets: The retained region approach. *Inf. Sci.*, 629:760–777, 2023.

[67] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[68] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[69] Jerry M. Mendel. Advances in type-2 fuzzy sets and systems. *Inf. Sci.*, 177:84–110, 2007.

[70] Jerry M. Mendel and Robert Ivor John. Type-2 fuzzy sets made simple. *IEEE Trans. Fuzzy Syst.*, 10:117–127, 2002.

[71] Ardashir Mohammadzadeh, Mohammad Hosein Sabzalian, and Weidong Zhang. An interval type-3 fuzzy system and a new online fractional-order learning algorithm: theory and practice. *IEEE Transactions on Fuzzy Systems*, 28(9):1940–1950, 2019.

[72] Deivanayagampillai Nagarajan, Malayalan Lathamaheswari, Said Broumi, and Jacob Kavikumar. A new perspective on traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets. *Operations Research Perspectives*, 2019.

[73] Zohreh Nazari and Batool Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16:173 – 185, 2018.

[74] Emanuel Ontiveros, Patricia Melin, and Oscar Castillo. Towards an efficient approach for mamdani interval type-3 fuzzy inference systems. *International Journal of Fuzzy Systems*, pages 1–19, 2024.

[75] Poli Venkatasubba Reddy. Type-n fuzzy control systems. Technical report, EasyChair, 2024.

[76] John T Rickard, Janet Aisbett, and Greg Gibbon. Fuzzy subsethood for fuzzy sets of type-2 and generalized type-n. *IEEE Transactions on Fuzzy Systems*, 17(1):50–60, 2008.

[77] John T Rickard, Janet Aisbett, and J Tyler Rickard. On a class of general type-n normal fuzzy sets synthesized from subject matter expert inputs. *IEEE Transactions on Fuzzy Systems*, 2024.

[78] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[79] Sovan Samanta, Madhumangal Pal, Hossein Rashmanlou, and Rajab Ali Borzooei. Vague graphs and strengths. *Journal of Intelligent & Fuzzy Systems*, 30(6):3675–3680, 2016.

[80] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[81] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[82] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[83] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[84] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[85] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[86] Florentin Smarandache. *Introduction to the Complex Refined Neutrosophic Set*. Infinite Study, 2017.

[87] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[88] Florentin Smarandache. Short introduction to standard and nonstandard neutrosophic set and logic. *Neutrosophic Sets and Systems*, 77:395–404, 2025.

[89] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[90] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[91] Smriti Srivastava and Rajesh Kumar. Design and application of a novel higher-order type-n fuzzy-logic-based system for controlling the steering angle of a vehicle: a soft computing approach. *Soft Computing*, 28(6):4743–4758, 2024.

[92] Li-Xin Wang. A new look at type-2 fuzzy sets and type-2 fuzzy logic systems. *IEEE Transactions on Fuzzy Systems*, 25:693–706, 2017.

[93] Dongrui Wu and Jerry M. Mendel. Similarity measures for closed general type-2 fuzzy sets: Overview, comparisons, and a geometric approach. *IEEE Transactions on Fuzzy Systems*, 27:515–526, 2019.

[94] Ting-Ting Xu and Jingjing Qin. A new representation method for type-2 fuzzy sets and its application to multiple criteria decision making. *International Journal of Fuzzy Systems*, 25:1171–1190, 2023.

[95] Hasan Yeti? and Mehmet Karakose. Modelling type-2 fuzzy systems by optimized nonstationary fuzzy sets with genetic algorithm. *2020 24th International Conference on Information Technology (IT)*, pages 1–4, 2020.

[96] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[97] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[98] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[99] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[100] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[101] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[102] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[103] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[104] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

[105] Mohammad Hossein Fazel Zarandi, R. Gamasaee, and Oscar Castillo. Type-1 to type-n fuzzy logic and systems. In *Fuzzy Logic in Its 50th Year*, 2016.

Chapter 10
Hyperfuzzy Hypersoft set and Hyperneutrosophic Hypersoft set

Takaaki Fujita [1] *
[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.

Abstract

Concepts such as Fuzzy Sets, Neutrosophic Sets, and Soft Sets are well-known for addressing uncertainty, with numerous applications explored in various fields. These concepts can be extended to Hyperfuzzy Sets, Hyperneutrosophic Sets, and Hypersoft Sets using hyperstructures (based on power sets).

In this paper, we define the Hyperneutrosophic Hypersoft Set and the Hyperfuzzy Hypersoft Set, and explore their mathematical structures and related properties.

Keywords: Neutrosophic Set, Hyperfuzzy set, Hypersoft set, Fuzzy Set, Soft Set, HyperNeutrosophic Set

1 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

1.1 Hyperfuzzy Set

Intuitively, hyperfuzzy Set extends the idea of fuzzy sets [71–79] into hierarchical structures, allowing for a more nuanced and flexible representation of uncertainty. The formal definition is provided below. A hyperfuzzy set generalizes the traditional fuzzy set framework [10, 17, 19, 25, 32, 33, 36, 41, 46, 50, 65].

Definition 1.1 (Set)**.** [31] A *set* is a well-defined collection of distinct objects, called *elements*. If x is an element of a set A, it is written as $x \in A$. Sets are typically represented using curly braces.

Definition 1.2. [71, 76] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Example 1.3. Let $Y = \{a, b, c, d\}$ represent a set of objects, e.g., four fruits:

$$Y = \{\text{Apple}, \text{Banana}, \text{Cherry}, \text{Date}\}.$$

Define a fuzzy set τ to represent the "degree of ripeness" of each fruit, with values assigned as follows:

$$\tau : Y \to [0, 1], \quad \tau(y) = \text{degree of ripeness of } y.$$

For example:

$$\tau(\text{Apple}) = 0.8, \quad \tau(\text{Banana}) = 0.6, \quad \tau(\text{Cherry}) = 0.9, \quad \tau(\text{Date}) = 0.3.$$

The fuzzy set τ can be expressed as:

$$\tau = \{(\text{Apple}, 0.8), (\text{Banana}, 0.6), (\text{Cherry}, 0.9), (\text{Date}, 0.3)\}.$$

Definition 1.4 (Powerset)**.** (cf. [15, 62]) The *powerset* of a set S, denoted as $\mathcal{P}(S)$, is the collection of all subsets of S, including the empty set and S itself:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.5 (HyperFuzzy Set)**.** [10, 19, 32, 41, 65] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1])$ is called a *hyperfuzzy set* over X, where $\tilde{P}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Example 1.6. Let $X = \{\text{Car}, \text{Bike}, \text{Bus}, \text{Train}\}$ represent a set of transport modes(cf. [3, 69]). Define a hyperfuzzy set $\tilde{\mu}$ to represent "levels of environmental friendliness" of each transport mode. Instead of assigning a single value, it assigns a subset of $[0, 1]$, reflecting uncertainty or variability in the assessment:

$$\tilde{\mu} : X \to \tilde{P}([0, 1]),$$

where $\tilde{P}([0, 1])$ denotes the collection of all non-empty subsets of $[0, 1]$. For example:

$$\tilde{\mu}(\text{Car}) = \{0.2, 0.4\}, \quad \tilde{\mu}(\text{Bike}) = \{0.8, 1.0\}, \quad \tilde{\mu}(\text{Bus}) = \{0.5, 0.7\}, \quad \tilde{\mu}(\text{Train}) = \{0.6, 0.9\}.$$

The hyperfuzzy set $\tilde{\mu}$ can be expressed as:

$$\tilde{\mu} = \{(\text{Car}, \{0.2, 0.4\}), (\text{Bike}, \{0.8, 1.0\}), (\text{Bus}, \{0.5, 0.7\}), (\text{Train}, \{0.6, 0.9\})\}.$$

1.2 Neutrosophic and HyperNeutrosophic Sets

Neutrosophic Sets extend Fuzzy Sets by introducing the concept of indeterminacy, which accounts for situations that are neither entirely true nor entirely false [34, 55–59, 63, 64]. As an advanced generalization, the HyperNeutrosophic Set has been developed, offering a more comprehensive framework for handling complex uncertainty [13, 16]. The relevant definitions are provided below.

Definition 1.7 (Neutrosophic Set)**.** [56] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Example 1.8. Let $X = \{\text{Paper}, \text{Plastic}, \text{Metal}, \text{Glass}\}$ represent a set of waste materials. Define a neutrosophic set A to represent the "recyclability" of each material (cf. [35, 43, 44]), characterized by three membership functions: $T_A(x)$ (truth), $I_A(x)$ (indeterminacy), and $F_A(x)$ (falsity). For example:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1].$$

Assign membership values as follows:

$$\begin{aligned}
&T_A(\text{Paper}) = 0.9, \quad I_A(\text{Paper}) = 0.05, \quad F_A(\text{Paper}) = 0.05,\\
&T_A(\text{Plastic}) = 0.6, \quad I_A(\text{Plastic}) = 0.2, \quad F_A(\text{Plastic}) = 0.2,\\
&T_A(\text{Metal}) = 0.8, \quad I_A(\text{Metal}) = 0.1, \quad F_A(\text{Metal}) = 0.1,\\
&T_A(\text{Glass}) = 0.7, \quad I_A(\text{Glass}) = 0.2, \quad F_A(\text{Glass}) = 0.1.
\end{aligned}$$

The neutrosophic set A can be expressed as:

$$A = \big\{(\text{Paper}, \langle 0.9, 0.05, 0.05\rangle), (\text{Plastic}, \langle 0.6, 0.2, 0.2\rangle), (\text{Metal}, \langle 0.8, 0.1, 0.1\rangle), (\text{Glass}, \langle 0.7, 0.2, 0.1\rangle)\big\}.$$

Definition 1.9 (HyperNeutrosophic Set)**.** [12, 16] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0, 1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Example 1.10. Let $X = \{\text{Paper}, \text{Plastic}, \text{Metal}, \text{Glass}\}$ represent the same set of waste materials. Define a hyperneutrosophic set $\tilde{A}$ to represent the "recyclability" of each material. Unlike the neutrosophic set, each membership value is a subset of $[0, 1]^3$, reflecting a range of possible truth (T), indeterminacy (I), and falsity (F) values. For example:

$$\tilde{\mu} : X \to \tilde{P}([0, 1]^3).$$

Assign membership subsets as follows:

$$\begin{aligned}\tilde{\mu}(\text{Paper}) &= \{(0.85, 0.05, 0.1), (0.9, 0.05, 0.05)\},\\ \tilde{\mu}(\text{Plastic}) &= \{(0.55, 0.2, 0.25), (0.6, 0.2, 0.2)\},\\ \tilde{\mu}(\text{Metal}) &= \{(0.75, 0.15, 0.1), (0.8, 0.1, 0.1)\},\\ \tilde{\mu}(\text{Glass}) &= \{(0.65, 0.25, 0.1), (0.7, 0.2, 0.1)\}.\end{aligned}$$

The hyperneutrosophic set $\tilde{A}$ can be expressed as:

$$\tilde{A} = \big\{(\text{Paper}, \{(0.85, 0.05, 0.1), (0.9, 0.05, 0.05)\}), (\text{Plastic}, \{(0.55, 0.2, 0.25), (0.6, 0.2, 0.2)\}),$$

$$(\text{Metal}, \{(0.75, 0.15, 0.1), (0.8, 0.1, 0.1)\}), (\text{Glass}, \{(0.65, 0.25, 0.1), (0.7, 0.2, 0.1)\})\big\}.$$

1.3 Hypersoft Set

A Soft Set offers a simplified framework for parameterized decision-making by mapping attributes or parameters to subsets of a universal set, effectively addressing uncertainty in a straightforward manner [7, 8, 20, 21, 37, 39, 42, 68]. Building on this concept, a Hypersoft Set enhances multi-attribute decision analysis by mapping combinations of multiple attributes to subsets of a universal set, allowing for a more nuanced and flexible approach [2, 4, 5, 14, 18, 27, 40, 49, 60].

A concise definition of the Hypersoft Set is provided below.

Definition 1.11 (Soft Set)**.** [37, 39] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Example 1.12. Let $U = \{\text{Car A}, \text{Car B}, \text{Car C}, \text{Car D}\}$ represent a universal set of cars, and let

$$A = \{\text{Color}, \text{Fuel Type}, \text{Price}\}$$

be a set of attributes. Define a soft set $(\mathcal{F}, S)$, where $S = \{\text{Color}, \text{Fuel Type}\} \subseteq A$, and $\mathcal{F} : S \to \mathcal{P}(U)$. The mapping $\mathcal{F}$ is given as:

$$\begin{aligned}\mathcal{F}(\text{Color}) &= \{\text{Car A}, \text{Car C}\},\\ \mathcal{F}(\text{Fuel Type}) &= \{\text{Car B}, \text{Car D}\}.\end{aligned}$$

The soft set $(\mathcal{F}, S)$ can be expressed as:

$$(\mathcal{F}, S) = \{(\text{Color}, \{\text{Car A}, \text{Car C}\}), (\text{Fuel Type}, \{\text{Car B}, \text{Car D}\})\}.$$

Here:

- $\mathcal{F}(\text{Color})$ represents the set of cars with a specific color.
- $\mathcal{F}(\text{Fuel Type})$ represents the set of cars with a specific fuel type.

Definition 1.13 (Hypersoft Set)**.** [60] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $C = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair (G, C), where $G : C \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, C) = \{(\gamma, G(\gamma)) \mid \gamma \in C, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in C$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Example 1.14. Let $U = \{\text{Car A}, \text{Car B}, \text{Car C}, \text{Car D}\}$ be the universal set of cars. Define attribute domains $\mathcal{A}_1 = \{\text{Red, Blue}\}$, $\mathcal{A}_2 = \{\text{Petrol, Diesel}\}$, and $\mathcal{A}_3 = \{\text{Expensive, Affordable}\}$. The Cartesian product of these domains is:

$$C = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3.$$

Define a hypersoft set (G, C), where $G : C \to \mathcal{P}(U)$. The mapping G is given as:

$$G(\text{Red}, \text{Petrol}, \text{Expensive}) = \{\text{Car A}\},$$
$$G(\text{Blue}, \text{Diesel}, \text{Affordable}) = \{\text{Car B}, \text{Car D}\},$$
$$G(\text{Red}, \text{Diesel}, \text{Affordable}) = \{\text{Car C}\}.$$

The hypersoft set (G, C) can be expressed as:

$$(G, C) = \{$$
$$((\text{Red}, \text{Petrol}, \text{Expensive}), \{\text{Car A}\}),$$
$$((\text{Blue}, \text{Diesel}, \text{Affordable}), \{\text{Car B}, \text{Car D}\}),$$
$$((\text{Red}, \text{Diesel}, \text{Affordable}), \{\text{Car C}\})\}.$$

Here:

- $G(\text{Red}, \text{Petrol}, \text{Expensive})$ represents the set of cars that are red, petrol-fueled, and expensive.
- $G(\text{Blue}, \text{Diesel}, \text{Affordable})$ represents the set of cars that are blue, diesel-fueled, and affordable.
- $G(\text{Red}, \text{Diesel}, \text{Affordable})$ represents the set of cars that are red, diesel-fueled, and affordable.

1.4 Fuzzy Hypersoft Set and Neutrosophic Hypersoft Set

A *Fuzzy Hypersoft Set* is a concept that combines the features of Fuzzy Sets and Hypersoft Sets, enabling it to handle uncertainties in multi-attribute decision-making scenarios [6,9,11,47,48,70]. Similarly, a *Neutrosophic Hypersoft Set* integrates the characteristics of Neutrosophic Sets and Hypersoft Sets, allowing it to address indeterminate and conflicting information in addition to uncertainties [1,26,30,38,45,51–54].

The definitions are presented below.

Definition 1.15 (Fuzzy Hypersoft Set (FHSS))**.** [11, 48, 70] Let K be a non-empty universal set, and let $t_1, t_2, t_3, \ldots, t_n$ be determining factors with distinct character traits corresponding to the sets $G_1, G_2, G_3, \ldots, G_n$, respectively. Assume that $G_i \cap G_j = \emptyset$ for $i \neq j$, where $i, j \in \{1, 2, \ldots, n\}$. Define $L = G_1 \times G_2 \times \cdots \times G_n$ as the Cartesian product of these sets. A *Fuzzy Hypersoft Set (FHSS)* over K is a pair (Θ, L), where:

$$\Theta : L \to \mathcal{P}(K)$$

is a mapping such that $\Theta(g) \subseteq K$ for every $g \in L$.

Here:

- $\Theta(g)$ represents a fuzzy subset of K corresponding to a combination $g \in L$.
- $L = G_1 \times G_2 \times \cdots \times G_n$ represents the Cartesian product of all determining factor subsets.

Example 1.16. Let

$$K = \{\text{Laptop 1}, \text{Laptop 2}, \text{Laptop 3}, \text{Laptop 4}\}$$

be a non-empty universal set, representing different laptop models(cf. [22, 24]). Suppose we have three determining factors with distinct character traits:

- t_1: *Brand* (with possible values $G_1 = \{\text{BrandA}, \text{BrandB}\}$),
- t_2: *CPU Type* (with possible values $G_2 = \{\text{Intel}, \text{AMD}\}$),

- t_3: *Price Category* (with possible values $G_3 = \{\text{Premium}, \text{Budget}\}$).

Assume each G_i is disjoint, i.e. $G_i \cap G_j = \emptyset$ for $i \neq j$. Define

$$L = G_1 \times G_2 \times G_3,$$

the Cartesian product of these sets. For instance,

$$L = \{ (\text{BrandA, Intel, Premium}), (\text{BrandB, AMD, Budget}), \dots \}.$$

A *Fuzzy Hypersoft Set* (Θ, L) over K is then defined by a mapping

$$\Theta : L \longrightarrow \mathcal{P}(K).$$

Concretely, let us define:

$$\begin{aligned}
\Theta(\text{BrandA}, \text{Intel}, \text{Premium}) &= \{\text{Laptop 1} \mapsto 0.9,\ \text{Laptop 2} \mapsto 0.2,\ \text{Laptop 3} \mapsto 0.1,\ \text{Laptop 4} \mapsto 0.0\},\\
\Theta(\text{BrandB}, \text{AMD}, \text{Budget}) &= \{\text{Laptop 1} \mapsto 0.0,\ \text{Laptop 2} \mapsto 0.7,\ \text{Laptop 3} \mapsto 0.4,\ \text{Laptop 4} \mapsto 0.2\},\\
\Theta(\text{BrandA}, \text{AMD}, \text{Budget}) &= \{\text{Laptop 1} \mapsto 0.3,\ \text{Laptop 2} \mapsto 0.1,\ \text{Laptop 3} \mapsto 0.8,\ \text{Laptop 4} \mapsto 0.6\},\\
&\text{etc. (for all other tuples in } L).
\end{aligned}$$

Here, each $\Theta(g)$ represents a fuzzy subset of K by assigning a membership degree in $[0, 1]$ to each laptop. For instance, $\Theta(\text{BrandA}, \text{Intel}, \text{Premium})(\text{Laptop 1}) = 0.9$ indicates *Laptop 1* matches "BrandA + Intel + Premium" with a fuzzy membership of 0.9.

Definition 1.17 (Neutrosophic Hypersoft Set (NHSS))**.** [51,52] Let ξ be the universal set, and let $P(\xi)$ denote the power set of ξ. Consider $l_1, l_2, l_3, \dots, l_n$ for $n \geq 1$, to be n-well-defined attributes. The corresponding attributive values are the sets $L_1, L_2, L_3, \dots, L_n$, where:

$$L_i \cap L_j = \emptyset, \quad \text{for } i \neq j,\ i, j \in \{1, 2, \dots, n\}.$$

Define the Cartesian product of these sets as:

$$L = L_1 \times L_2 \times \cdots \times L_n.$$

A *Neutrosophic Hypersoft Set (NHSS)* over ξ is a pair $(\mathcal{F}, L)$, where:

$$\mathcal{F} : L \to P(\xi),$$

is a mapping such that:

$$\mathcal{F}(L) = \{\langle x, T(\mathcal{F}(L)), I(\mathcal{F}(L)), F(\mathcal{F}(L))\rangle \mid x \in \xi\},$$

where:

- $T : \xi \to [0, 1]$, $I : \xi \to [0, 1]$, $F : \xi \to [0, 1]$ are membership functions representing:
  - T: Truth membership value,
  - I: Indeterminacy membership value,
  - F: Falsity membership value.
- For each $x \in \xi$, the membership values satisfy:

$$0 \leq T(\mathcal{F}(L)) + I(\mathcal{F}(L)) + F(\mathcal{F}(L)) \leq 3.$$

Example 1.18. Let

$$\xi = \{\text{Customer A}, \text{Customer B}, \text{Customer C}, \text{Customer D}\}$$

be the universal set representing different customers. Suppose we define three well-defined attributes:

- l_1: *Payment Method* with possible values $L_1 = \{\text{CreditCard}, \text{Cash}\}$,

- l_2: *Delivery Speed* with possible values $L_2 = \{\text{Express}, \text{Standard}\}$,
- l_3: *Promotion Type* with possible values $L_3 = \{\text{Discount}, \text{Coupon}\}$.

Assume $L_i \cap L_j = \emptyset$ for $i \neq j$. Define

$$L \;=\; L_1 \times L_2 \times L_3.$$

A *Neutrosophic Hypersoft Set* $(\mathcal{F}, L)$ over ξ is then defined by a mapping

$$\mathcal{F} : L \;\longrightarrow\; \mathcal{P}(\xi),$$

and for each $A \in L$,

$$\mathcal{F}(A) \;=\; \Big\{\langle x,\ T(\mathcal{F}(A)),\ I(\mathcal{F}(A)),\ F(\mathcal{F}(A))\rangle \mid x \in \xi\Big\},$$

where $T, I, F : \xi \to [0, 1]$ are truth, indeterminacy, and falsity membership functions satisfying $0 \leq T+I+F \leq 3$.

For instance, let us define:

$$\mathcal{F}(\text{CreditCard}, \text{Express}, \text{Discount}) = \big\{\langle \text{Customer A},\ 0.8,\ 0.1,\ 0.1\rangle,\ \langle \text{Customer B},\ 0.2,\ 0.3,\ 0.5\rangle, \ldots\big\},$$
$$\mathcal{F}(\text{Cash}, \text{Standard}, \text{Coupon}) = \big\{\langle \text{Customer A},\ 0.0,\ 0.7,\ 0.3\rangle,\ \langle \text{Customer C},\ 0.5,\ 0.2,\ 0.3\rangle, \ldots\big\},$$
$$\text{etc. (for all other tuples in } L).$$

Here,

- $\mathcal{F}(\text{CreditCard}, \text{Express}, \text{Discount})(\text{Customer A})$ indicates *Customer A* has truth membership 0.8, indeterminacy membership 0.1, and falsity membership 0.1 under the attribute-combination

$$(\text{CreditCard}, \text{Express}, \text{Discount})$$

.

- The sum $T + I + F$ for each customer in each combination does not exceed 3.

2 Results of This Paper

In this paper, we propose new definitions for various types of sets and briefly examine their relationships with existing concepts.

2.1 HyperFuzzy Hypersoft Set (HFHSS)

A *HyperFuzzy Hypersoft Set (HNHSS)* is a set-theoretic concept that combines the principles of HyperFuzzy Sets and Hypersoft Sets. The definition and related concepts are presented below.

Definition 2.1 (HyperFuzzy Hypersoft Set (HFHSS))**.** Let K be a non-empty universal set, and let $t_1, t_2, \ldots, t_n$ be determining factors with distinct sets of attributive values $G_1, G_2, \ldots, G_n$, each pairwise disjoint. Define the Cartesian product

$$L = G_1 \times G_2 \times \cdots \times G_n.$$

A *HyperFuzzy Hypersoft Set (HFHSS)* over K is a pair (Υ, L), where:

$$\Upsilon : L \;\longrightarrow\; \widetilde{\mathcal{P}}([0, 1]),$$

and for each $\ell \in L$, $\Upsilon(\ell) \subseteq [0, 1]$. In other words, the mapping Υ assigns a *set of membership degrees* for each multi-attribute combination $\ell \in L$.

In more detail:

- For each $\ell = (\ell_1, \ell_2, \ldots, \ell_n) \in L$, $\Upsilon(\ell)$ is a *hyperfuzzy subset* of K, meaning that each $x \in K$ is associated with a *set* of membership values $\subseteq [0,1]$, reflecting the uncertainty or variability in membership.
- $\ell_i \in G_i$ indicates the chosen value for the i-th attribute.

Example 2.2. Suppose we have a universal set

$$K = \{\text{Mobile A}, \text{Mobile B}, \text{Mobile C}\}$$

representing different smartphone models(cf. [23]). Let us consider two determining factors (attributes), each with distinct sets of possible values:

- t_1 : *Camera Quality* with possible values $G_1 = \{\text{BasicCam}, \text{ProCam}\}$,
- t_2 : *Memory Size* with possible values $G_2 = \{\text{64GB}, \text{128GB}, \text{256GB}\}$.

Assume $G_1 \cap G_2 = \emptyset$. Define the Cartesian product

$$L = G_1 \times G_2 = \{ (\text{BasicCam}, \text{64GB}),\ (\text{ProCam}, \text{128GB}),\ (\text{ProCam}, \text{256GB}), \ldots \}.$$

A *HyperFuzzy Hypersoft Set* (Υ, L) over K is then defined by a mapping

$$\Upsilon : L \longrightarrow \widetilde{\mathcal{P}}([0,1]),$$

which assigns to each multi-attribute combination $\ell \in L$ a *set of membership degrees* in the interval $[0,1]$. Concretely, let us specify a few values of Υ:

$$\begin{aligned}
\Upsilon(\text{BasicCam}, \text{64GB}) &= \{\text{Mobile A} \mapsto \{0.2, 0.5\}, \quad \text{Mobile B} \mapsto \{0.1\}, \quad \text{Mobile C} \mapsto \{0.0, 0.3, 0.7\}\}, \\
\Upsilon(\text{ProCam}, \text{128GB}) &= \{\text{Mobile A} \mapsto \{0.6\}, \quad \text{Mobile B} \mapsto \{0.8, 0.9\}, \quad \text{Mobile C} \mapsto \{0.1, 0.2\}\}, \\
\Upsilon(\text{ProCam}, \text{256GB}) &= \{\text{Mobile A} \mapsto \{0.4\}, \quad \text{Mobile B} \mapsto \{0.5, 0.6, 0.7\}, \quad \text{Mobile C} \mapsto \{0.0\}\},
\end{aligned}$$

and so forth for any other pair $(\ell_1, \ell_2) \in L$.

Here, for example, $\Upsilon(\text{BasicCam}, \text{64GB})(\text{Mobile A}) = \{0.2, 0.5\}$ means that *Mobile A* belongs to the attribute combination *(BasicCam,64GB)* with a *set* of membership degrees $\{0.2, 0.5\} \subseteq [0,1]$, reflecting uncertainty or variability in how strongly *Mobile A* matches those attributes.

Theorem 2.3. *The following holds.*

(1) HFHSS extends Fuzzy Hypersoft Set: If each set of membership degrees in $\Upsilon(\ell)$ is restricted to be a singleton in $[0,1]$, then an HFHSS reduces to an FHSS.

(2) HFHSS extends Hyperfuzzy Set: If $n = 1$ and we collapse the multi-attribute domain G_1 to a single attribute, then an HFHSS reduces to a Hyperfuzzy Set for that attribute's domain.

Proof. Part (1). Consider an HFHSS (Υ, L). By definition, each $\Upsilon(\ell)$ is a set of membership values. Impose the constraint:

$$\Upsilon(\ell) = \{\mu_\ell\} \subseteq [0,1], \quad \text{a singleton for each } \ell.$$

Hence, each multi-attribute combination ℓ yields exactly one membership degree in $[0,1]$. This is precisely the structure of a Fuzzy Hypersoft Set, where the mapping from $\ell \in L$ to a single membership function $\mu_\ell : K \to [0,1]$. Therefore, under that restriction, an HFHSS merges into an FHSS.

Part (2). Let $n = 1$. The domain $L = G_1$ has a single attribute with values in G_1. Then the pair (Υ, G_1) assigns to each $g \in G_1$ a subset $\Upsilon(g) \subseteq [0,1]$, i.e., a hyperfuzzy subset. But this is nothing but a Hyperfuzzy Set $\widetilde{\mu} : G_1 \to \widetilde{\mathcal{P}}([0,1])$ if we identify $G_1 \equiv X$. Conclusively, HFHSS generalizes the single-attribute Hyperfuzzy concept. □

2.2 HyperNeutrosophic Hypersoft Set (HNHSS)

A *HyperNeutrosophic Hypersoft Set (HNHSS)* is a set-theoretic concept that combines the principles of HyperNeutrosophic Sets and Hypersoft Sets. The definition and related concepts are presented below.

Definition 2.4 (HyperNeutrosophic Hypersoft Set (HNHSS))**.** Let ξ be a universal set, and let $\{l_1, l_2, \ldots, l_n\}$ be n-well-defined attributes with attributive values $\{L_1, L_2, \ldots, L_n\}$, each pairwise disjoint. Define

$$L = L_1 \times L_2 \times \cdots \times L_n.$$

A *HyperNeutrosophic Hypersoft Set (HNHSS)* over ξ is a pair

$$\Big(\Gamma, L\Big),$$

where

$$\Gamma : L \longrightarrow \widetilde{\mathcal{P}}\big([0,1]^3\big).$$

Hence, for each $\ell \in L$, $\Gamma(\ell) \subseteq [0,1]^3$ is a *set of neutrosophic membership triplets* $\big(T, I, F\big)$ with $0 \le T+I+F \le 3$. This extends the notion of a Neutrosophic Hypersoft Set by allowing hyper-subsets of membership triplets rather than single membership triplets.

Concretely,

- Each $\ell \in L$ corresponds to a combination of attributive values.
- $\Gamma(\ell) \subseteq [0,1]^3$ indicates possible (T, I, F) triplets for each $x \in \xi$. Typically, one can think of $\Gamma(\ell)$ as a *hyperneutrosophic* mapping that assigns multiple (T, I, F)-vectors, reflecting further uncertainty or gradation of truth, indeterminacy, and falsity.

Example 2.5. Let

$$\xi = \{\text{Customer 1}, \text{Customer 2}, \text{Customer 3}\}$$

represent different customers. Suppose we have two attributes:

- l_1 : *Payment Option* with $L_1 = \{\text{Credit}, \text{Debit}\}$,
- l_2 : *Shipping Mode* with $L_2 = \{\text{Express}, \text{Regular}\}$.

Define the Cartesian product

$$L = L_1 \times L_2 = \{ (\text{Credit}, \text{Express}), (\text{Credit}, \text{Regular}), (\text{Debit}, \text{Express}), (\text{Debit}, \text{Regular})\}.$$

A *HyperNeutrosophic Hypersoft Set* (Γ, L) over ξ is then defined by

$$\Gamma : L \longrightarrow \widetilde{\mathcal{P}}\big([0,1]^3\big),$$

where for each $(\ell_1, \ell_2) \in L$, $\Gamma(\ell_1, \ell_2) \subseteq [0,1]^3$ is a *set of* (T, I, F) triplets with $0 \le T + I + F \le 3$. Concretely, let us illustrate:

$$\begin{aligned}
\Gamma(\text{Credit}, \text{Express}) = \big\{ &\text{Customer 1} \mapsto \{(0.8, 0.1, 0.1), (0.5, 0.3, 0.2)\}, \\
&\text{Customer 2} \mapsto \{(0.0, 0.7, 0.3)\}, \\
&\text{Customer 3} \mapsto \{(0.4, 0.3, 0.3)\}\big\},
\end{aligned}$$

$$\begin{aligned}
\Gamma(\text{Debit}, \text{Regular}) = \big\{ &\text{Customer 1} \mapsto \{(0.6, 0.2, 0.2)\}, \quad \text{Customer 2} \mapsto \{(0.2, 0.5, 0.3), (0.3, 0.4, 0.3)\}, \\
&\text{Customer 3} \mapsto \{(0.1, 0.1, 0.8)\}\big\},
\end{aligned}$$

and so on for the other combinations.

For instance, $\Gamma(\text{Credit}, \text{Express})(\text{Customer 1}) = \{(0.8, 0.1, 0.1), (0.5, 0.3, 0.2)\}$ indicates two possible neutrosophic memberships for *Customer 1* under the combination *(Credit,Express)*. Each triplet (T, I, F) satisfies $T + I + F \le 3$. This structure extends the standard Neutrosophic Hypersoft Set to a hyper-level, capturing multiple (T, I, F)-membership evaluations under each multi-attribute scenario.

Theorem 2.6. *The following holds.*

(1) HNHSS extends Neutrosophic Hypersoft Set: If each subset of neutrosophic membership triplets is restricted to a single element, the HNHSS reduces to an NHSS.

(2) HNHSS extends HyperNeutrosophic Set: If $n = 1$ and we collapse the domain L_1 to a single attribute, the HNHSS reduces to a standard HyperNeutrosophic Set, where each element is mapped to a set of (T, I, F)-triplets.

Proof. Part (1). Suppose we have an HNHSS (Γ, L). By definition, each $\Gamma(\ell) \subseteq [0, 1]^3$ is a set of possible neutrosophic triplets. Impose the restriction:

$$\Gamma(\ell) = \{ (T_\ell, I_\ell, F_\ell) \} \subseteq [0, 1]^3, \quad \text{a singleton.}$$

Hence, for each $\ell \in L$, we have exactly one triplet (T_ℓ, I_ℓ, F_ℓ). This is precisely the structure of a (single-valued) *Neutrosophic Hypersoft Set*, in which the triple (T, I, F) satisfies $0 \le T + I + F \le 3$. Therefore, under that restriction, HNHSS collapses to NHSS.

Part (2). Let $n = 1$. Then the set of attributive values is L_1, so the Cartesian product $L = L_1$. The mapping $\Gamma : L_1 \to \widetilde{\mathcal{P}}([0, 1]^3)$ essentially assigns to each $g \in L_1$ a set of triplets (T, I, F). This is exactly a *HyperNeutrosophic Set* if we identify $g \in L_1$ with each element of the original universe. Conclusively, HNHSS generalizes the single-attribute hyperneutrosophic concept. □

Theorem 2.7. *Let (Γ, L) be a* HyperNeutrosophic Hypersoft Set *over a universe ξ, as in Definition. If we require that the* indeterminacy *coordinate I is always zero within every neutrosophic membership subset $\Gamma(\ell) \subseteq [0, 1]^3$, then (Γ, L) reduces to a* HyperFuzzy Hypersoft Set *(HFHSS). Conversely, any HFHSS can be embedded into a HyperNeutrosophic Hypersoft framework by assigning zero indeterminacy.*

Proof. Let (Γ, L) be a HyperNeutrosophic Hypersoft Set (HNHSS). By definition, each $\Gamma(\ell)$ is a non-empty *set* of neutrosophic membership triplets

$$(T, I, F) \in [0, 1]^3 \quad \text{with} \quad 0 \le T + I + F \le 3.$$

Consider imposing the additional condition that *indeterminacy is always zero*:

$$\forall (T, I, F) \in \Gamma(\ell), \quad I = 0.$$

Under this restriction, each triplet $(T, 0, F)$ must still satisfy $T + F \le 3$. In fact, for many classical fuzzy or semi-fuzzy contexts, we often require $T + F \le 1$. Even if we allow up to ≤ 3, one can re-normalize or interpret T as a membership subset in $[0, 1]$. Concretely:

- Since $I = 0$, the only meaningful degrees are T (representing membership) and F (representing non-membership).
- If $\Gamma(\ell) \subseteq \{(T, 0, F) : T, F \in [0, 1], T + F \le 1\}$, we directly get a *hyperfuzzy* subset in $[0, 1]$ by identifying each $(T, 0, F)$ with the single membership value T (and possibly some interpretation for F).
- As a result, each element of $\Gamma(\ell)$ can be seen as a *set of membership degrees* in $[0, 1]$, which is precisely the structure of an HFHSS (see Definition 2.1).

Thus, under the no-indeterminacy constraint $I = 0$, the HyperNeutrosophic Hypersoft Set (Γ, L) degenerates into a HyperFuzzy Hypersoft Set.

Conversely, suppose (Υ, L) is a *HyperFuzzy Hypersoft Set*, where each $\Upsilon(\ell) \subseteq [0, 1]$. We embed Υ into a HyperNeutrosophic Hypersoft Set by defining

$$\Gamma(\ell) = \Big\{ (T, 0, F) \in [0, 1]^3 : T \in \Upsilon(\ell),\ F = 1 - T \text{ (or similar)} \Big\}.$$

Hence we set the indeterminacy coordinate to zero and interpret membership T from the original HFHSS. The resulting structure $\Gamma : L \to \tilde{\mathcal{P}}([0, 1]^3)$ is a valid hyperneutrosophic mapping, thus embedding any HFHSS into an HNHSS. This shows the *strict generalization*: HNHSS extends HFHSS.

Hence, *HyperNeutrosophic Hypersoft Sets* strictly generalize *HyperFuzzy Hypersoft Sets*. □

3 Future Tasks of this Research

Extensions utilizing concepts such as the SuperHypersoft Set [61, 80], Hypersoft Rough Set [66, 67], and Hypersoft Expert Set [28, 29] are considered as potential future directions for this research.

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Shereen .. and Mahmoud M. Ismail. Interval-valued neutrosophic hypersoft sets (ivnhss) for enterprise resource planning selection. *Journal of Neutrosophic and Fuzzy Systems*, 2023.

[2] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[3] Taqi Abrar. Physical activity calculation by transport modes detection using fuzzy logic. *Journal of traffic and transportation engineering (Valley Cottage, NY)*, 9(2):e2–e2, 2021.

[4] Nehmat Ahmed and Osama T. Pirbal. Plithogenic crisp hypersoft topology. *European Journal of Pure and Applied Mathematics*, 2024.

[5] Mohammed Abdullah Al-Hagery, Abdalla I Abdalla Musa, et al. Automated credit card risk assessment using fuzzy parameterized neutrosophic hypersoft expert set. *International Journal of Neutrosophic Science*, (1):93–3, 2025.

[6] Ashraf Al-Quran, Faisal Al-Sharqi, Kifayat Ullah, Mamika Ujianita Romdhini, Marwa Balti, and Mohammed Alomair. *Bipolar fuzzy hypersoft set and its application in decision making*. Infinite Study, 2023.

[7] José Carlos R Alcantud, Azadeh Zahedi Khameneh, Gustavo Santos-García, and Muhammad Akram. A systematic literature review of soft set theory. *Neural Computing and Applications*, 36(16):8951–8975, 2024.

[8] M Irfan Ali, Feng Feng, Xiaoyan Liu, Won Keun Min, and Muhammad Shabir. On some new operations in soft set theory. *Computers & Mathematics with Applications*, 57(9):1547–1553, 2009.

[9] Muhammad Arshad, Muhammad Saeed, Atiqe Ur Rahman, Mazin Abed Mohammed, Karrar Hameed Abdulkareem, Jan Nedoma, Radek Martinek, and Muhammet Deveci. A robust framework for the selection of optimal covid-19 mask based on aggregations of interval-valued multi-fuzzy hypersoft sets. *Expert systems with applications*, 238:121944, 2024.

[10] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[11] Somen Debnath. Fuzzy hypersoft sets and its weightage operator for decision making. *Journal of fuzzy extension and applications*, 2(2):163–170, 2021.

[12] Takaaki Fujita. Short survey on the hierarchical uncertainty of fuzzy, neutrosophic, and plithogenic sets.

[13] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[14] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[15] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[16] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[17] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[18] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[19] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[20] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[21] Ke Gong, Panpan Wang, and Zhi Xiao. Bijective soft set decision system based parameters reduction under fuzzy environments. *Applied Mathematical Modelling*, 37:4474–4485, 2013.

[22] Shankha S Goswami and Dhiren K Behera. An overview of multiple criteria decision making techniques in the selection of best laptop model. *Advances in Systems Science and Applications*, 23(2):11–23, 2023.

[23] Shankha Shubhra Goswami and Dhiren Kumar Behera. Evaluation of the best smartphone model in the market by integrating fuzzy-ahp and promethee decision-making approach. *Decision*, 48:71–96, 2021.

[24] Shankha Shubhra Goswami, Dhiren Kumar Behera, and Soupayan Mitra. A comprehensive study of weighted product model for selecting the best laptop model available in the market. *Brazilian Journal of Operations & Production Management*, 17(2):1–18, 2020.

[25] Areej Tawfeeq Hameed, Rusul Ahmed Flayyih, and Saba Hussein Ali. Bipolar hyper fuzzy at-ideals of at-algebra.

[26] R. Hema, R. Sudharani, and M. Kavitha. A novel approach on plithogenic interval valued neutrosophic hypersoft sets and its application in decision making. *Indian Journal Of Science And Technology*, 2023.

[27] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Hypersoft expert set with application in decision making for recruitment process. 2021.

[28] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Single valued neutrosophic hypersoft expert set with application in decision making. 2021.

[29] Muhammad Ihsan, Muhammad Haris Saeed, and Atiqe Ur Rahman. An intuitionistic fuzzy hypersoft expert set-based robust decision-support framework for human resource management integrated with modified topsis and correlation coefficient. *Neural Computing and Applications*, pages 1–25, 2023.

[30] Muhammad Naveed Jafar, Muhammad Haris Saeed, Kainat Muniba Khan, Faten S. Alamri, and Hamiden A. Wahed Khalifa. Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. *IEEE Access*, PP:1–1, 2022.

[31] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[32] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[33] Young Bae Jun, Seon Jeong Kim, and Seok Zun Song. Hyper permeable values and energetic sets in bck/bci-algebras. 2020.

[34] M Kaviyarasu, Muhammad Aslam, Farkhanda Afzal, Maha Mohammed Saeed, Arif Mehmood, and Saeed Gul. The connectivity indices concept of neutrosophic graph and their application of computer network, highway system and transport network flow. *Scientific Reports*, 14(1):4891, 2024.

[35] Vassilis S Kouikoglou and Yannis A Phillis. A monotonic fuzzy system for assessing material recyclability. In *2005 IEEE International Conference on Systems, Man and Cybernetics*, volume 4, pages 3426–3431. IEEE, 2005.

[36] M MAHARIN. Hyper fuzzy cosets. *Scholar: National School of Leadership*, 9(1.2), 2020.

[37] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[38] Mesfer Ahmed Mesfer. Blockchain with single-valued neutrosophic hypersoft sets assisted threat detection for secure iot assisted consumer electronics. *International Journal of Neutrosophic Science*.

[39] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[40] Sagvan Y Musa and Baravan A Asaad. Topological structures via bipolar hypersoft sets. *Journal of Mathematics*, 2022(1):2896053, 2022.

[41] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[42] S. Onar. A note on neutrosophic soft set over hyperalgebras. *Symmetry*, 16(10):1288, 2024.

[43] Yannis A Phillis, Vassilis S Kouikoglou, and Xiaomin Zhu. A fuzzy logic approach to the evaluation of material recyclability. In *The 14th IEEE International Conference on Fuzzy Systems, 2005. FUZZ'05.*, pages 454–458. IEEE, 2005.

[44] Yannis A Phillis, Vassilis S Kouikoglou, and Xiaomin Zhu. Fuzzy assessment of material recyclability and its applications. *Journal of Intelligent and Robotic Systems*, 55:21–38, 2009.

[45] Atiqe Ur Rahman, Muhammad Haris Saeed, and Hamiden Abd El-Wahed Khalifa. Multi-attribute decision-making based on aggregations and similarity measures of neutrosophic hypersoft sets with possibility setting. *Journal of Experimental & Theoretical Artificial Intelligence*, 36:161 – 186, 2022.

[46] Belhamdi Saad, Chekroun Salim, and Djalal Eddine Khodja. Application of hyper-fuzzy logic type-2 in field oriented control of induction motor with broken bars. *IOSR Journal of Engineering (IOSRJEN)*, 8(5):1–7, 2018.

[47] Muhammad Saeed, Muhammad Ahsan, and Thabet Abdeljawad. A development of complex multi-fuzzy hypersoft set with application in mcdm based on entropy and similarity measure. *IEEE Access*, 9:60026–60042, 2021.

[48] Muhammad Saeed and Muhammad Imran Harl. Fundamentals of picture fuzzy hypersoft set with application. *Neutrosophic Sets and Systems*, 53(1):24, 2023.

[49] Muhammad Saeed, Muhammad Khubab Siddique, Muhammad Ahsan, Muhammad Rayees Ahmad, and Atiqe Ur Rahman. A novel approach to the rudiments of hypersoft graphs. *Theory and Application of Hypersoft Set, Pons Publication House, Brussel*, pages 203–214, 2021.

[50] Omar M Salim, MA Zohdy, HT Dorrah, and AM Kamel. Application of hyper-fuzzy logic in field oriented control of induction machines. In *Proceedings of 14th International Middle East Power Systems Conference*, pages 356–363, 2010.

[51] Muhammad Saqlain, Naveed Jafar, Sana Moin, Muhammad Farhan Saeed, and Said Broumi. Single and multi-valued neutrosophic hypersoft set and tangent similarity measure of single valued neutrosophic hypersoft sets. *Neutrosophic Sets and Systems*, 32:20, 2020.

[52] Muhammad Saqlain, Sana Moin, Muhammad Naveed Jafar, Muhammad Saeed, and Florentin Smarandache. *Aggregate operators of neutrosophic hypersoft set*. Infinite Study, 2020.

[53] Muhammad Saqlain and Xiao Long Xin. *Interval valued, m-polar and m-polar interval valued neutrosophic hypersoft sets*. Infinite Study, 2020.

[54] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[55] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[56] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[57] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[58] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[59] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[60] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[61] Florentin Smarandache. Foundation of the superhypersoft set and the fuzzy extension superhypersoft set: A new vision. *Neutrosophic Systems with Applications*, 11:48–51, 2023.

[62] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[63] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[64] Florentin Smarandache and AA Salama. Neutrosophic crisp set theory. 2015.

[65] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[66] V. S. Subha and R. Selvakumar. An innovative electric vehicle selection with multi-criteria decision-making approach in indian brands on a neutrosophic hypersoft rough set by using reduct and core. *2024 Fourth International Conference on Advances in Electrical, Computing, Communication and Sustainable Technologies (ICAECT)*, pages 1–7, 2024.

[67] V. S. Subha and R. Selvakumar. A novel approach to plithogenic neutrosophic hypersoft rough set and its application to decision making problem with reference to bakery industry. *Communications in Mathematics and Applications*, 2024.

[68] Varun Kumar Tiwari, Prashant Kumar Jain, and Puneet Tandon. An integrated shannon entropy and topsis for product design concept evaluation based on bijective soft set. *Journal of Intelligent Manufacturing*, 30:1645 – 1658, 2017.

[69] Umut R Tuzkaya and Semih Önüt. A fuzzy analytic network process based approach to transportation-mode selection between turkey and germany: A case study. *Information Sciences*, 178(15):3133–3146, 2008.

[70] Adem Yolcu and Taha Yasin Ozturk. Fuzzy hypersoft sets and it's application to decision-making. *Theory and application of hypersoft set*, 50, 2021.

[71] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[72] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[73] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[74] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[75] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[76] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[77] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[78] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[79] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

[80] Yue Zhao. Enhanced exprom methodology using superhypersoft sets for evaluating the quality of cultural industry development in the digital economy with single-valued neutrosophic sets. *Neutrosophic Sets and Systems*, 79:342–354, 2025.

Chapter 11

Short Review of SuperFuzzy, SuperNeutrosophic, and SuperPlithogenic Set

Takaaki Fujita [1] *

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.

Abstract

Concepts such as Fuzzy Sets [21,43], Neutrosophic Sets [29,30], and Plithogenic Sets [33] have been extensively studied to address uncertainty, with diverse applications across numerous fields. Building upon the Plithogenic Set, the HyperPlithogenic Set and SuperHyperPlithogenic Set have also gained significant attention [16].

In this paper, we examine SuperFuzzy Sets, Super-Intuitionistic Fuzzy Sets, Super-Neutrosophic Sets, Super-Quadripartitioned Neutrosophic Sets, Super-Pentapartitioned Neutrosophic Sets, Super-Heptapartitioned Neutrosophic Sets, and SuperPlithogenic Sets. This work serves as a reconsideration and extension of studies such as those in [16, 32].

Keywords: Plithogenic Set, HyperPlithogenic Set, n-SuperhyperPlithogenic Set, Plithogenic Cubic Set

1 Preliminaries and Definitions

This section introduces the fundamental concepts and definitions necessary for the discussions and analyses presented in this paper.

1.1 Plithogenic Set and Uncertain Set

The Plithogenic Set provides a mathematical structure designed to accommodate multi-valued degrees of membership and contradiction, enabling its effective application to complex decision-making scenarios. Its properties and potential applications have been extensively studied in works such as [1, 14, 26–28, 36, 39].

Further extending its utility, related constructs like the Plithogenic Graph have also been examined in various contexts [12, 18]. The Plithogenic Set serves as a unifying framework, generalizing numerous existing mathematical constructs, including:

- *Fuzzy Sets*: Developed as a means to handle imprecision and uncertainty [43, 44].
- *Intuitionistic Fuzzy Sets*: Extending Fuzzy Sets by incorporating degrees of non-membership [2, 4].
- *Vague Sets*: A framework with two bounds for truth and falsehood [7, 19].
- *Neutrosophic Sets*: Introducing degrees of truth, indeterminacy, and falsehood [30, 31].
- *Picture Fuzzy Sets*: Incorporating positive, negative, and neutral opinions [9, 25, 38].
- *Bipolar Neutrosophic Sets*: Allowing positive and negative degrees of truth [11, 42].
- *Hesitant Fuzzy Sets*: Accounting for multiple potential membership values [40, 41].

The ability of the Plithogenic Set to encapsulate these diverse frameworks highlights its adaptability and broad applicability in modeling uncertainty and multi-valued reasoning.

The definitions of existing concepts, including Fuzzy Sets, Intuitionistic Fuzzy Sets, Neutrosophic Sets, Quadripartitioned Neutrosophic Sets, Pentapartitioned Neutrosophic Sets, Heptapartitioned Neutrosophic Sets, and Plithogenic Sets, are presented below.

Definition 1.1 (Base Set)**.** A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 1.2 (Powerset)**.** [15, 24] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the collection of all possible subsets of S, including both the empty set and S itself. Formally, it is expressed as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 1.3 (Fuzzy set)**.** [43, 45] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 1.4 (Intuitionistic Fuzzy Set)**.** [2–4] An *Intuitionistic Fuzzy Set (IFS)* A in a universe of discourse X is defined as a set of ordered pairs:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X, \mu_A(x), \nu_A(x) \in [0, 1], \mu_A(x) + \nu_A(x) \leq 1\},$$

where:

- $\mu_A(x)$ is the *degree of membership* of x in A,
- $\nu_A(x)$ is the *degree of non-membership* of x in A,
- $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ represents the *degree of indeterminacy or hesitation*.

Definition 1.5 (Neutrosophic Set)**.** [31] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 1.6 (Quadripartitioned Neutrosophic Set (QNS))**.** (cf. [8, 10, 20]) Let X be a universe of discourse. A *Quadripartitioned Neutrosophic Set (QNS)* on X is given by

$$QNS = \{\langle x, T(x), C(x), U(x), F(x)\rangle \mid x \in X\},$$

where each of $T(x), C(x), U(x), F(x)$ lies in $[0, 1]$, satisfying

$$0 \quad \leq \quad T(x) + C(x) + U(x) + F(x) \quad \leq \quad 4.$$

Definition 1.7 (Pentapartitioned Neutrosophic Set (PNS))**.** (cf. [5, 22]) Let X be a universe of discourse. A *Pentapartitioned Neutrosophic Set (PNS)* on X is given by

$$PNS = \{\langle x, T(x), C(x), R(x), U(x), F(x)\rangle \mid x \in X\},$$

where each of $T(x), C(x), R(x), U(x), F(x) \in [0, 1]$, satisfying

$$0 \quad \leq \quad T(x) + C(x) + R(x) + U(x) + F(x) \quad \leq \quad 5.$$

Definition 1.8 (Heptapartitioned Neutrosophic Set)**.** [6, 23] A Heptapartitioned Neutrosophic Set (HNS) on a universe X is defined as:

$$HNS = \{\langle x, T(x), C(x), R(x), U(x), F(x), G(x), L(x)\rangle \mid x \in X\},$$

where $T(x), C(x), R(x), U(x), F(x), G(x), L(x) \in [0, 1]$, and

$$0 \leq T(x) + C(x) + R(x) + U(x) + F(x) + G(x) + L(x) \leq 7.$$

Definition 1.9 (Plithogenic Set)**.** [34,35] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set* PS is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* [1]
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a)$$

2 Results in This Paper

This paper presents the results derived through the study conducted herein. It serves as a reconsideration of the work discussed in [32].

2.1 SuperFuzzy, SuperIntuitionistic Fuzzy, and SuperNeutrosophic Sets

The definitions of SuperFuzzy, SuperIntuitionistic Fuzzy, and SuperNeutrosophic Sets are provided below [32]. These concepts are extensions of the Fuzzy Set, Intuitionistic Fuzzy Set, and Neutrosophic Set, respectively.

Additionally, related concepts such as the SuperHyperFuzzy Set and SuperHyperNeutrosophic Set are also known in the literature [13, 16, 17, 32].

Definition 2.1. [32] A *Superfuzzy Set* is defined as a function:

$$\tau : P(A) \to [0,1],$$

where $P(A)$ is the powerset of a non-empty set A, and $\tau(S)$ for $S \in P(A)$ represents the degree of membership (truth) of the subset S in A.

Example 2.2. For example:

$$\tau(\{a_1, a_2\}) = 0.9$$

indicates that the subset $\{a_1, a_2\}$ as a whole has a membership degree of 0.9 in A.

[1] It is important to note that the definition of the Degree of Appurtenance Function varies across different papers. Some studies define this concept using the power set, while others simplify it by avoiding the use of the power set [37]. The author has consistently defined the Classical Plithogenic Set without employing the power set.

Example 2.3 (Superfuzzy Set for Group Travel Plans)**.** Consider a set A of possible tourist destinations:

$$A = \{\text{Paris}, \text{Rome}, \text{Berlin}\}.$$

The powerset $P(A)$ contains all possible *subsets* of destinations, such as {Paris, Rome} or {Berlin}, etc.

We define a superfuzzy set τ to capture how *feasible* each combination of destinations is within a certain travel budget. Let us assume:

$$\begin{aligned}
&\tau(\emptyset) = 0.0 \quad \text{(no destination is not considered feasible)},\\
&\tau(\{\text{Paris}\}) = 0.9 \quad \text{(Paris alone is quite feasible)},\\
&\tau(\{\text{Rome}\}) = 0.7 \quad \text{(Rome alone is moderately feasible)},\\
&\tau(\{\text{Berlin}\}) = 0.8,\\
&\tau(\{\text{Paris}, \text{Rome}\}) = 0.5,\\
&\tau(\{\text{Paris}, \text{Berlin}\}) = 0.6,\\
&\tau(\{\text{Rome}, \text{Berlin}\}) = 0.4,\\
&\tau(\{\text{Paris}, \text{Rome}, \text{Berlin}\}) = 0.2.
\end{aligned}$$

Here, $\tau(\{\text{Paris}, \text{Rome}\}) = 0.5$ means that traveling to both Paris and Rome together has a 0.5 feasibility degree under the budget constraints, while the subset {Paris, Rome, Berlin} has an even lower feasibility (0.2).

Definition 2.4. [32] A *Superintuitionistic Fuzzy Set* is defined as a function:

$$\tau : P(A) \to [0, 1]^2,$$

where $P(A)$ is the powerset of a non-empty set A, and $\tau(S) = (\mu(S), \nu(S))$ for $S \in P(A)$ represents:

- $\mu(S)$: degree of membership (truth) of the subset S,
- $\nu(S)$: degree of non-membership (falsehood) of the subset S.

These values satisfy:

$$\mu(S) + \nu(S) \leq 1.$$

Example 2.5. For example:

$$\tau(\{a_1, a_2\}) = (0.5, 0.7)$$

indicates $\mu(\{a_1, a_2\}) = 0.5$ and $\nu(\{a_1, a_2\}) = 0.7$.

Example 2.6 (Superintuitionistic Fuzzy Set for Joint Purchase Decision)**.** Suppose $A = \{\text{Laptop}, \text{Tablet}\}$. Think of A as potential tech items a family might buy. The powerset $P(A)$ includes $\emptyset$, {Laptop}, {Tablet}, and {Laptop, Tablet}.

We define a superintuitionistic fuzzy set τ to represent:

$$\tau(S) = (\mu(S),\ \nu(S)),$$

where $\mu(S)$ is the "confidence that the subset S is a good purchase choice," and $\nu(S)$ is the "confidence that S is a poor choice." They satisfy $\mu(S) + \nu(S) \leq 1$. For instance:

$$\begin{aligned}
&\tau(\emptyset) = (0.0, 0.8) \quad \big(\text{no purchase is almost certainly a bad idea: } \mu = 0, \nu = 0.8\big),\\
&\tau(\{\text{Laptop}\}) = (0.6, 0.2), \quad \text{(fairly good idea; somewhat not)},\\
&\tau(\{\text{Tablet}\}) = (0.4, 0.3), \quad \text{(mixed feelings about just a tablet)},\\
&\tau(\{\text{Laptop}, \text{Tablet}\}) = (0.7, 0.2).
\end{aligned}$$

In $\tau(\{\text{Laptop}\}) = (0.6, 0.2)$, we see the family's membership is 0.6 (a fairly good choice) and non-membership is 0.2 (some negative doubts). These sum to 0.8, which is ≤ 1.

Definition 2.7. [32] A *Superneutrosophic Set* is defined as a function:

$$\tau : P(A) \to [0, 1]^3,$$

where $P(A)$ is the powerset of a non-empty set A, and $\tau(S) = (T(S), I(S), F(S))$ for $S \in P(A)$ represents:

- $T(S)$: degree of truth of the subset S,
- $I(S)$: degree of indeterminacy of the subset S,
- $F(S)$: degree of falsehood of the subset S.

These values satisfy:

$$0 \le T(S) + I(S) + F(S) \le 3.$$

Example 2.8. For example:

$$\tau(\{a_1, a_2\}) = (0.8, 0.1, 0.3)$$

indicates $T(\{a_1, a_2\}) = 0.8$, $I(\{a_1, a_2\}) = 0.1$, and $F(\{a_1, a_2\}) = 0.3$.

Example 2.9 (Superneutrosophic Set for Project Collaborations)**.** Let $A = \{\text{Team A}, \text{Team B}, \text{Team C}\}$ represent potential teams. We examine subsets of A for a hypothetical *collaboration project*.

Define $\tau(S) = (T(S), I(S), F(S))$:

$$\begin{aligned}
\tau(\emptyset) &= (0.0,\ 0.3,\ 0.0), && \text{(no team chosen: uncertain but not "false" nor "true")},\\
\tau(\{\text{Team A}\}) &= (0.8,\ 0.0,\ 0.1), && \text{(very likely a good partner, low falsehood, no major indeterminacy)},\\
\tau(\{\text{Team B}\}) &= (0.6,\ 0.3,\ 0.2), && \text{(some uncertainty about B, moderate truth, small falsehood)},\\
\tau(\{\text{Team C}\}) &= (0.2,\ 0.7,\ 0.1), && \text{(C is mostly unknown, so high indeterminacy)},\\
\tau(\{\text{Team A}, \text{Team B}\}) &= (0.7,\ 0.2,\ 0.5), && \text{(cooperation might be more complicated; sum = 1.4)},\\
\tau(\{\text{Team A}, \text{Team C}\}) &= (0.6,\ 0.9,\ 0.4), && \text{(lots of unknowns; sum = 1.9)},\\
\tau(\{\text{Team B}, \text{Team C}\}) &= (0.5,\ 0.7,\ 0.6), && \text{(still feasible but uncertain; sum = 1.8)},\\
\tau(\{\text{Team A}, \text{Team B}, \text{Team C}\}) &= (0.9,\ 0.3,\ 0.8).
\end{aligned}$$

For example, $\tau(\{\text{Team A}, \text{Team C}\}) = (0.6, 0.9, 0.4)$ implies we see a *relatively positive* truth-value of 0.6, yet a *high indeterminacy* (0.9), and a *somewhat large falsehood* (0.4). The sum $= 1.9 \le 3$ remains within allowed bounds.

2.2 Super-Quadripartitioned, Pentapartitioned, Heptapartitioned Neutrosophic Set

The definitions of Super-Quadripartitioned, Super-Pentapartitioned, and Super-Heptapartitioned Neutrosophic Sets are presented below.

Definition 2.10. A *Super-Quadripartitioned Neutrosophic Set (Super-QNS)* on X is:

$$\text{Super-QNS} = \{\langle S, T(S), C(S), U(S), F(S)\rangle \mid S \in P(X)\},$$

where $T(S), C(S), U(S), F(S) \in [0, 1]$ satisfy

$$0 \quad \le \quad T(S) + C(S) + U(S) + F(S) \quad \le \quad 4.$$

Example 2.11 (Super-QNS Example)**.** Let $X = \{x_1, x_2\}$. We list several subsets $S \subseteq X$ with illustrative values of $\big(T(S), C(S), U(S), F(S)\big)$. All values lie in $[0, 1]$, so their sum is automatically ≤ 4.

$$\begin{aligned}
&\langle\, \emptyset,\ 0.0,\ 0.0,\ 0.0,\ 0.0\rangle \quad (\text{sum} = 0.0),\\
&\langle\, \{x_1\},\ 0.4,\ 0.2,\ 0.1,\ 0.3\rangle \quad (\text{sum} = 1.0),\\
&\langle\, \{x_2\},\ 0.8,\ 0.1,\ 0.1,\ 0.7\rangle \quad (\text{sum} = 1.7),\\
&\langle\, \{x_1, x_2\},\ 0.9,\ 0.9,\ 0.9,\ 0.9\rangle \quad (\text{sum} = 3.6).
\end{aligned}$$

Hence, the corresponding Super-QNS (restricted to these four subsets) is:

$$\text{Super-QNS} \;=\; \big\{\langle \emptyset, 0.0, 0.0, 0.0, 0.0\rangle,\ \langle \{x_1\}, 0.4, 0.2, 0.1, 0.3\rangle,\ \langle \{x_2\}, 0.8, 0.1, 0.1, 0.7\rangle,\ \langle \{x_1, x_2\}, 0.9, 0.9, 0.9, 0.9\rangle\big\}.$$

Definition 2.12. A *Super-Pentapartitioned Neutrosophic Set (Super-PNS)* on X is:

$$\text{Super-PNS} = \{\langle S, T(S), C(S), R(S), U(S), F(S)\rangle \mid S \in P(X)\},$$

where $T(S), C(S), R(S), U(S), F(S) \in [0, 1]$ satisfy

$$0 \;\leq\; T(S) + C(S) + R(S) + U(S) + F(S) \;\leq\; 5.$$

Example 2.13 (Super-PNS Example)**.** Let $X = \{x_1, x_2\}$. Below, we assign a 5-tuple $(T(S), C(S), R(S), U(S), F(S))$ for some subsets S. Since each value is in $[0, 1]$, the total sum is ≤ 5.

$$\begin{aligned}
&\langle\, \emptyset,\ 0.0,\ 0.0,\ 0.0,\ 0.0,\ 0.0\rangle \quad (\text{sum} = 0.0),\\
&\langle\, \{x_1\},\ 0.2,\ 0.2,\ 0.2,\ 0.1,\ 0.4\rangle \quad (\text{sum} = 1.1),\\
&\langle\, \{x_1, x_2\},\ 0.9,\ 0.8,\ 0.3,\ 0.3,\ 1.0\rangle \quad (\text{sum} = 3.3).
\end{aligned}$$

Hence, a partial Super-PNS on X (for these three subsets) is:

$$\text{Super-PNS} \;=\; \Bigl\{\langle\emptyset, 0, 0, 0, 0, 0\rangle,\ \langle\{x_1\}, 0.2, 0.2, 0.2, 0.1, 0.4\rangle,\ \langle\{x_1, x_2\}, 0.9, 0.8, 0.3, 0.3, 1.0\rangle\Bigr\}.$$

Definition 2.14. A *Super-Heptapartitioned Neutrosophic Set (Super-HNS)* on X is:

$$\text{Super-HNS} = \{\langle S, T(S), C(S), R(S), U(S), F(S), G(S), L(S)\rangle \mid S \in P(X)\},$$

where $T(S), C(S), R(S), U(S), F(S), G(S), L(S) \in [0, 1]$ satisfy

$$0 \;\leq\; T(S) + C(S) + R(S) + U(S) + F(S) + G(S) + L(S) \;\leq\; 7.$$

Example 2.15 (Super-HNS Example)**.** Again, let $X = \{x_1, x_2\}$. We exhibit three subsets, each with a 7-tuple (T, C, R, U, F, G, L) in $[0, 1]$:

$$\begin{aligned}
&\langle\, \emptyset,\ 0, 0, 0, 0, 0, 0, 0\rangle \quad (\text{sum} = 0),\\
&\langle\, \{x_1\},\ 0.5,\ 0.6,\ 0.1,\ 0.9,\ 0.0,\ 0.2,\ 0.0\rangle \quad (\text{sum} = 2.3),\\
&\langle\, \{x_1, x_2\},\ 1.0,\ 0.8,\ 0.7,\ 0.7,\ 0.4,\ 0.9,\ 0.5\rangle \quad (\text{sum} = 5.0).
\end{aligned}$$

Thus a partial Super-HNS (covering these subsets) is:

$$\text{Super-HNS} = \Bigl\{\langle\emptyset,\ 0, 0, 0, 0, 0, 0, 0\rangle,\ \langle\{x_1\},\ 0.5, 0.6, 0.1, 0.9, 0.0, 0.2, 0.0\rangle,\ \langle\{x_1, x_2\},\ 1.0, 0.8, 0.7, 0.7, 0.4, 0.9, 0.5\rangle\Bigr\}.$$

2.3 SuperPlithogenic Set

The definition of the SuperPlithogenic Set is provided below.

Definition 2.16 (SuperPlithogenic Set)**.** Let X be a universal set, and let $P(X)$ denote its powerset. Consider an *attribute* v, whose possible values range over some set P_v. Define:

$$pdf : P(X) \times P_v \longrightarrow [0, 1]^s, \quad pCF : P_v \times P_v \longrightarrow [0, 1]^t,$$

where s and t are positive integers determined by context (e.g., how many dimensions of membership or contradiction we wish to encode). We require that for all $a, b \in P_v$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a, a) = 0,$$

2. *Symmetry of Contradiction Function*:
$$pCF(a, b) = pCF(b, a).$$

Then a *SuperPlithogenic Set (SPS)* on X is defined by the 5-tuple

$$\text{SPS} = \Big(P(X),\ v,\ P_v,\ pdf,\ pCF\Big).$$

Remark 2.17. The key difference from the classical Plithogenic Set [34,35] is that the *degree of appurtenance function* pdf now takes as input a subset $S \subseteq X$ (instead of a single element $x \in X$) and an attribute value $a \in P_v$. Hence, any membership, truth, or contradiction measure can be assigned to a *subset* of X, rather than to a single element.

Theorem 2.18. *Let* SPS $= (P(X), v, P_v, pdf, pCF)$ *be any SuperPlithogenic Set. Suppose we only evaluate* pdf *on* singleton subsets $\{x\} \subseteq X$. *Then the restriction of* SPS *to singletons coincides with a classical plithogenic set*

$$PS = (X, v, P_v, pdf', pCF),$$

where

$$pdf'(x, a) \ = \ pdf\big(\{x\}, a\big).$$

Proof. By assumption, the SuperPlithogenic structure is SPS $= (P(X), v, P_v, pdf, pCF)$. Consider the restricted domain $\{\{x\} \mid x \in X\}$, i.e., the collection of singleton subsets of X. Define

$$pdf'(x, a) := pdf(\{x\}, a).$$

Because pdf was originally defined for all subsets $S \in P(X)$, it is in particular defined for singletons $\{x\}$. Hence pdf' is well-defined as a function

$$pdf' : X \times P_v \ \longrightarrow \ [0,1]^s.$$

The contradiction function pCF remains unchanged; it was already defined to satisfy reflexivity and symmetry over P_v. Thus the 4-tuple (X, v, P_v, pdf', pCF) satisfies the original axioms for a (classical) Plithogenic Set [34].

Therefore, restricting to singletons recovers the usual Plithogenic Set structure, demonstrating that the classical Plithogenic Set is indeed a special case (or subsystem) of the SuperPlithogenic Set. □

Theorem 2.19. *Let* $(P(X), v, P_v, pdf, pCF)$ *be any SuperPlithogenic Set with* $s = 1$ *and ignore* pCF *by setting it identically to zero (or to any constant function if desired). Define*

$$\tau(S) := pdf(S, a_0)$$

for some chosen $a_0 \in P_v$. *Then* $\tau : P(X) \to [0,1]$ *becomes a Superfuzzy Set.*

Proof. Since $s = 1$, the output of $pdf(S, a)$ is a single real value in $[0,1]$. Pick any specific attribute value $a_0 \in P_v$. Define

$$\tau(S) \ = \ pdf(S, a_0).$$

By construction,

$$\tau : \ P(X) \ \to \ [0,1].$$

Hence, by Definition of a Superfuzzy Set, τ assigns to each subset $S \subseteq X$ a single membership degree in $[0,1]$. If we do not utilize the contradiction function pCF (e.g., set it identically to zero), the structure is effectively capturing only "membership" degrees for subsets. Thus τ is precisely a Superfuzzy Set as in the usual sense (cf. the definition in the question). □

Theorem 2.20. *Let* SPS $= (P(X), v, P_v, pdf, pCF)$ *be a SuperPlithogenic Set with output dimension* $s = 1$ *(for simplicity). Define for each* $S \subseteq X$ *and* $a \in P_v$,

$$\mu(S, a) := pdf(S, a) \ \in \ [0,1].$$

Suppose we have a continuous t-norm $\star$ *and t-conorm* $\diamond$ *on* $[0,1]$. *Define new functions*

$$\mu_\cup(S, a) \ = \ \mu\big(S_1 \cup S_2,\ a\big) \ = \ \mu(S_1, a) \ \diamond \ \mu(S_2, a),$$

$$\mu_\cap(S, a) \ = \ \mu\big(S_1 \cap S_2,\ a\big) \ = \ \mu(S_1, a) \ \star \ \mu(S_2, a).$$

Then the operations $\cup$ *and* $\cap$ *induce well-defined membership values under* $\star$ *and* $\diamond$*, yielding a* plithogenic-based union and intersection *for subsets inside the SuperPlithogenic framework.*

Proof. Step 1. Well-Definition.
Given two subsets $S_1, S_2 \subseteq X$ and any $a \in P_v$, we define

$$\mu(S_1 \cup S_2, a) = \mu(S_1, a) \diamond \mu(S_2, a),$$

$$\mu(S_1 \cap S_2, a) = \mu(S_1, a) \star \mu(S_2, a).$$

Since $\star$ and $\diamond$ are by definition binary operations on $[0, 1]$ that remain within $[0, 1]$, the results are indeed in $[0, 1]$.

Step 2. Consistency with T-norm/T-conorm Properties.
T-norms $\star$ and T-conorms $\diamond$ satisfy properties such as commutativity, associativity, monotonicity, and identity element existence. Hence, if we extend them to membership values of subsets (in the SuperPlithogenic sense), the set-theoretic union and intersection remain consistent with the underlying fuzzy or plithogenic interpretations.

Step 3. Conclusion.
Thus, any pair of subsets in $P(X)$ can be combined via $\cup, \cap$ in a manner that is *internally* consistent with the membership assignments generated by the plithogenic-based function $\mu(\cdot, a)$. This shows that union and intersection can be interpreted in the SuperPlithogenic environment using t-norms and t-conorms, ensuring the standard fuzzy-lattice properties carry over. □

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Walid Abdelfattah. Variables selection procedure for the dea overall efficiency assessment based plithogenic sets and mathematical programming. *International Journal of Scientific Research and Management*, 2022.

[2] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[3] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[4] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[5] Pranab Biswas, Surapati Pramanik, and Bibhas Chandra Giri. Single valued bipolar pentapartitioned neutrosophic set and its application in madm strategy. 2022.

[6] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[7] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[8] Rajashi Chatterjee, Pinaki Majumdar, and Syamal Kumar Samanta. A multi-criteria group decision making algorithm with quadripartitioned neutrosophic weighted aggregation operators using quadripartitioned neutrosophic numbers in ipqsvnss environment. *Soft Computing*, 24(12):8857–8880, 2020.

[9] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[10] Somen Debnath. Fuzzy quadripartitioned neutrosophic soft matrix theory and its decision-making approach. *Journal of Computational and Cognitive Engineering*, 1(2):88–93, 2022.

[11] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[12] Takaaki Fujita. Review of plithogenic directed, mixed, bidirected, and pangene offgraph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 120.

[13] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[14] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 74, 2024.

[15] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[16] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[17] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[18] Takaaki Fujita. A short note on the basic graph construction algorithm for plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 274, 2025.

[19] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[20] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[21] Hongxing Li and Vincent C Yen. *Fuzzy sets and fuzzy decision-making*. CRC press, 1995.

[22] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[23] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[24] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[25] Muhammad Saeed and Muhammad Imran Harl. Fundamentals of picture fuzzy hypersoft set with application. *Neutrosophic Sets and Systems*, 53(1):24, 2023.

[26] S.Gomathy, Deivanayagampillai Nagarajan, Said Broumi, and M.Lathamaheswari. Plithogenic sets and their application in decision making plithogenic sets and their application in decision making. 2020.

[27] Prem Kumar Singh. Plithogenic set for multi-variable data analysis. *International Journal of Neutrosophic Science*, 2020.

[28] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Science*, 2022.

[29] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[30] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[31] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[32] Florentin Smarandache. Hyperuncertain, superuncertain, and superhyperuncertain sets/logics/probabilities/statistics. *Critical Review*, XIV, 2017.

[33] Florentin Smarandache. *Plithogeny, plithogenic set, logic, probability, and statistics*. Infinite Study, 2017.

[34] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[35] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[36] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[37] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[38] Libor Svadlenka, Vladimir Simic, Momcilo Dobrodolac, Dragana Lazarevic, and Gordana Todorovic. Picture fuzzy decision-making approach for sustainable last-mile delivery. *IEEE Access*, 8:209393–209414, 2020.

[39] Devendra Kumar Tayal, Sumit Kumar Yadav, and Divya Arora. Personalized ranking of products using aspect-based sentiment analysis and plithogenic sets. *Multimedia Tools and Applications*, 82:1261–1287, 2022.

[40] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[41] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[42] Vakkas Ulucay, Irfan Deli, and Mehmet Sahin. Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making. *Neural Computing and Applications*, 29:739–748, 2018.

[43] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[44] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[45] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

Chapter 12
Superhypersoft Rough set, Superhypersoft Expert set, and Bipolar Superhypersoft Set

Takaaki Fujita [1] *
[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.

Abstract

Soft sets provide a robust framework for decision-making by mapping parameters to subsets of a universal set, addressing uncertainty and vagueness. This paper explores advanced extensions of soft sets, including six novel concepts: Superhypersoft Rough Set, Superhypersoft Expert Set, Bipolar Superhypersoft Set, Treesoft Rough Set, Treesoft Expert Set, and Bipolar Treesoft Set, alongside their definitions and properties.

Keywords: Superhypersoft set, Rough set, Soft Set, Treesoft set, Hypersoft set

1 Short Introduction of this paper

1.1 Soft Sets and Rough Sets

Soft sets are mathematical tools designed for decision-making, providing a framework that maps parameters to subsets of a universal set, effectively handling uncertainty and vagueness [77, 79]. Over the years, various extensions of soft sets have been developed, including Hypersoft Sets [107], Superhypersoft Sets [109, 118], Treesoft Sets [20,84], Fuzzy Soft Sets [19,98,125], Neutrosophic Soft Sets [13,14,17], Soft Expert Sets [6,102], and Soft Rough Sets [34] (which integrate rough set [43, 85–92] theory with soft sets), as well as Bipolar Soft Sets. These extensions have been extensively studied and applied in numerous fields.

This paper focuses on Superhypersoft Sets and Treesoft Sets.

- *Hypersoft Sets* extend soft sets by mapping combinations of multiple attributes to subsets of a universal set, thereby enhancing multi-attribute decision-making capabilities [2, 61, 107].
- *Superhypersoft Sets* further generalize hypersoft sets by mapping power set combinations of multiple attribute values to subsets of a universal set, enabling higher-dimensional decision-making frameworks [40, 67, 109, 116, 118].
- *Treesoft Sets* employ hierarchical attribute trees, mapping nodes and leaves of the tree structure to subsets of a universal set, offering a structured approach for analyzing complex hierarchical data [20, 84].

1.2 Our Contribution in This Paper

In this paper, we introduce and analyze the following concepts: Superhypersoft Rough Set, Superhypersoft Expert Set, Bipolar Superhypersoft Set, Treesoft Rough Set, Treesoft Expert Set, and Bipolar Treesoft Set. Each of these represents an extension of previously established set concepts. We anticipate that these advanced structures will inspire further research, particularly in applications such as decision-making.

2 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper.

2.1 SuperHypersoft Set and Treesoft Set

This subsection introduces the concepts of Soft Sets, Hypersoft Sets, Treesoft Sets, and SuperHypersoft Sets, which serve as foundational tools for advanced decision-making frameworks. A Soft Set provides a simplified approach to parameterized decision modeling by mapping attributes, or parameters, to subsets of a universal set, thereby addressing uncertainty in a straightforward manner [55,77,79]. Expanding on this idea, a Hypersoft Set facilitates enhanced multi-attribute decision analysis by mapping combinations of multiple attributes to subsets of a universal set [2, 5, 36, 50, 61, 83, 99, 107]. Treesoft Sets provide a structured framework for analyzing hierarchical data. By utilizing hierarchical attribute trees, Treesoft Sets map both nodes and leaves of the tree to subsets of a universal set, allowing for a detailed and organized representation of complex hierarchical relationships [111]. Further generalizing the principles of Hypersoft Sets, SuperHypersoft Sets extend their functionality by mapping power set combinations of multiple attribute values to subsets of a universal set. This approach enables higher-dimensional decision-making and accommodates complex interrelationships among attributes [40, 41, 52, 67, 75, 109, 114–116, 118, 142]. The theoretical underpinnings and definitions of these sets will be discussed in detail.

Definition 2.1 (Soft Set)**.** [77,79] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 2.2 (Hypersoft Set)**.** [107] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $C = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair (G, C), where $G : C \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, C) = \{(\gamma, G(\gamma)) \mid \gamma \in C, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in C$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Definition 2.3 (SuperHyperSoft Set)**.** [109] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$C = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair (F, C), where:

$$F : C \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in C$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, C) = \{(\gamma, F(\gamma)) \mid \gamma \in C, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in C$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Definition 2.4. [110] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as $\text{Tree}(A)$, with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where $\text{Tree}(A)$ represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

2.2 Hypersoft Rough Set

A hypersoft rough set uses lower and upper approximations over an approximation space to represent multi-attribute uncertainty in sets. Note that an approximation space is a mathematical structure that models uncertainty, consisting of a universe of objects and an equivalence relation, enabling the definition of lower and upper approximations for subsets.

Definition 2.5 (Pawlak Approximation Space)**.** [126, 127] Let U be a non-empty set, known as the *universe of discourse*, and let R be an equivalence relation on U. The pair (U, R) is called a *Pawlak approximation space*. The relation R induces a partition of U into equivalence classes, where each element in U is indiscernible from others within its equivalence class.

Given a subset $X \subseteq U$, the *lower approximation* $\underline{R}(X)$ and the *upper approximation* $\overline{R}(X)$ of X with respect to R are defined as follows:

- The *lower approximation* of X, denoted $\underline{R}(X)$, is the set of all elements $u \in U$ such that the equivalence class $[u]_R \subseteq X$:
$$\underline{R}(X) = \{u \in U \mid [u]_R \subseteq X\}.$$
This set contains elements that are certainly in X based on the information provided by R.
- The *upper approximation* of X, denoted $\overline{R}(X)$, is the set of all elements $u \in U$ such that the intersection $[u]_R \cap X \neq \emptyset$:
$$\overline{R}(X) = \{u \in U \mid [u]_R \cap X \neq \emptyset\}.$$
This set contains elements that possibly belong to X given the information provided by R.

The pair $(\underline{R}(X), \overline{R}(X))$ represents the rough set approximation of X within the Pawlak approximation space (U, R).

Definition 2.6 (Soft Rough Set)**.** (cf. [1, 4, 31, 71, 74, 78, 104, 137]) Let U be a universal set, A a set of parameters, and $P(U)$ the power set of U. Let R be an equivalence relation on U, inducing a partition $U/R = \{Y_1, Y_2, \ldots, Y_m\}$ into equivalence classes. A soft set (F, A) on U is defined as $F : A \to P(U)$.

For $B \subseteq U$, the *Soft Rough Lower Approximation* $L(B)$ and *Soft Rough Upper Approximation* $U(B)$ are given by:

$$L(B) = \{u \in U \mid \exists e \in A \text{ such that } F(e) \subseteq B\},$$
$$U(B) = \{u \in U \mid \exists e \in A \text{ such that } F(e) \cap B \neq \emptyset\}.$$

The *Soft Rough Set* is represented as the pair:

$$(L(B), U(B)),$$

where $L(B)$ and $U(B)$ are the approximations of B with respect to the soft set.

Definition 2.7 (Hypersoft Rough Set)**.** [69,120,121] Let (X, R) be a Pawlak approximation space, where R is an equivalence relation on X. Given a Hypersoft Set (F, J) over X, the *Hypersoft Lower Approximation* $F_*(\mathbf{j})$ and *Hypersoft Upper Approximation* $F^*(\mathbf{j})$ of F with respect to R are defined for each $\mathbf{j} \in J$ as:

$$F_*(\mathbf{j}) = \{x \in X \mid [x]_R \subseteq F(\mathbf{j})\},$$

$$F^*(\mathbf{j}) = \{x \in X \mid [x]_R \cap F(\mathbf{j}) \neq \emptyset\},$$

where $[x]_R$ denotes the equivalence class of x under R.

The *Hypersoft Rough Set* is then the pair (F_*, F^*, J).

2.3 Hypersoft Expert Set

The following outlines the concept of the Soft Expert Set [6–9, 11, 12, 16, 23, 58, 100–102] and its extension, the Hypersoft Expert Set [61–66].

Definition 2.8 (Soft Expert Set)**.** (cf. [8, 9, 11, 100]) Let U be a universal set, E a set of parameters, X a set of experts, and $O = \{0, 1\}$ a set of opinions. Define $Z = E \times X \times O$ and $A \subseteq Z$.

A *Soft Expert Set* over U is a pair (F, A), where:

- $F : A \to \mathcal{P}(U)$ is a mapping that assigns each $\alpha \in A$ a subset of U, with $\mathcal{P}(U)$ denoting the power set of U.

The Soft Expert Set is represented as:

$$(F, A) = \{(\alpha, F(\alpha)) \mid \alpha \in A, F(\alpha) \subseteq U\}.$$

Definition 2.9. (cf. [62–64]) A *Hypersoft Expert Set (HSE-Set)* is a pair (Ψ, S) over a universe of discourse Ω, where:

- $\Psi : S \to \mathcal{P}(\Omega)$ is a mapping such that $\Psi(a, d, c) \subseteq \Omega$, where $S \subseteq T = G \times D \times C$, and:
  - $G = G_1 \times G_2 \times \cdots \times G_n$ is the Cartesian product of n disjoint attributive sets $G_1, G_2, \ldots, G_n$, corresponding to attributes $g_1, g_2, \ldots, g_n$.
  - D is a set of specialists (decision-makers).
  - C is a set of conclusions, typically $C = \{0 \text{ (disagree)}, 1 \text{ (agree)}\}$.
- Elements of S take the form (a, d, c), where:
  - $a \in G$ represents an attribute combination (e.g., $a = (g_{11}, g_{21}, g_{31})$).
  - $d \in D$ represents a specialist.
  - $c \in C$ represents a conclusion.

2.4 Bipolar Hypersoft Set

A Bipolar Soft Set represents positive and negative memberships, ensuring consistency by mapping parameters to subsets of a universal set [10, 26, 27, 59, 72, 122]. A Bipolar Hypersoft Set extends Bipolar Soft Sets by incorporating multi-attribute combinations for positive and negative memberships in decision-making frameworks [22, 80–82].

Definition 2.10 (Bipolar Soft Set)**.** [26,27,122] A *Bipolar Soft Set* over a universal set U is a triple (F, G, A), where:

- $F : A \to P(U)$ is the *positive membership mapping*,

- $G : \neg A \to P(U)$ is the *negative membership mapping*,
- $A \subseteq E$, $\neg A = E \setminus A$, where E is a set of parameters.

The mappings satisfy the *consistency constraint*:

$$F(e) \cap G(\neg e) = \emptyset, \quad \forall e \in A.$$

A Bipolar Soft Set is represented as:

$$(F, G, A) = \{(e, F(e), G(\neg e)) \mid e \in A, F(e) \cap G(\neg e) = \emptyset\}.$$

Definition 2.11 (Bipolar Hypersoft Set)**.** [80–82] A *Bipolar Hypersoft Set (BHS-Set)* is a triple (F, G, A) over a universe of discourse U, where:

- $F : A \to \mathcal{P}(U)$ and $G : \neg A \to \mathcal{P}(U)$, with $\mathcal{P}(U)$ denoting the power set of U.
- The mappings satisfy the *consistency constraint*:

$$F(\alpha) \cap G(\neg\alpha) = \emptyset, \quad \forall \alpha \in A.$$

- $A = A_1 \times A_2 \times \cdots \times A_n$, where $A_i \subseteq E_i$ and $E = E_1 \times E_2 \times \cdots \times E_n$.
- $\neg A = \neg A_1 \times \neg A_2 \times \cdots \times \neg A_n$, where $\neg A_i = E_i \setminus A_i$.

The BHS-Set (F, G, A) is represented as:

$$(F, G, A) = \{(\alpha, F(\alpha), G(\neg\alpha)) \mid \alpha \in A \text{ and } F(\alpha) \cap G(\neg\alpha) = \emptyset\}.$$

3 Results in This Paper

This paper introduces six new set concepts as significant results.

3.1 SuperHypersoft Rough Set

The SuperHypersoft Rough Set, which generalizes the Hypersoft Rough Set, is presented below.

Definition 3.1 (SuperHypersoft Rough Set)**.** Let U be a non-empty universe and (U, R) be a Pawlak approximation space, where R is an equivalence relation on U. Suppose there are n distinct attributes $a_1, \ldots, a_n$, each having a set of possible values A_i such that $A_i \cap A_j = \varnothing$ for all $i \neq j$. Define the Cartesian product of the power sets:

$$C = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A *SuperHypersoft Set* is a mapping

$$F : C \longrightarrow \mathcal{P}(U).$$

For each $\gamma \in C$, define the *superhypersoft lower approximation* $F_*(\gamma)$ and the *superhypersoft upper approximation* $F^*(\gamma)$ as follows:

$$F_*(\gamma) = \{u \in U \mid [u]_R \subseteq F(\gamma)\}, \quad F^*(\gamma) = \{u \in U \mid [u]_R \cap F(\gamma) \neq \varnothing\}.$$

Here, $[u]_R$ denotes the equivalence class of u under R. The *SuperHypersoft Rough Set* is the triple

$$(F_*, F^*, C),$$

where $F_*, F^* : C \to \mathcal{P}(U)$ are the lower- and upper-approximation mappings, respectively.

Theorem 3.2 (Generalization Property of SuperHypersoft Rough Set)**.** *A SuperHypersoft Rough Set generalizes both the Hypersoft Rough Set and the SuperHypersoft Set.*

Proof. We show the reduction of the SuperHypersoft Rough Set to each simpler case:

(i) Generalizing the Hypersoft Rough Set. In a classical Hypersoft Rough Set, each attribute domain is A_i, and we form $A_1 \times A_2 \times \cdots \times A_n$ as the product of these domains. By restricting every $\mathcal{P}(A_i)$ to consist only of singletons, we effectively recover A_i rather than $\mathcal{P}(A_i)$. Hence $C = \mathcal{P}(A_1) \times \cdots \times \mathcal{P}(A_n)$ collapses to $A_1 \times \cdots \times A_n$. In that scenario, $F(\gamma)$ matches the usual Hypersoft mapping. The induced lower/upper approximations $F_*(\gamma), F^*(\gamma)$ then coincide with the definitions of a Hypersoft Rough Set.

(ii) Generalizing the SuperHypersoft Set. If R is taken to be the identity relation on U, then each equivalence class $[u]_R$ is just $\{u\}$. It follows that

$$F_*(\gamma) = \{\, u \in U : [u]_R \subseteq F(\gamma)\} = \{\, u \in U : \{u\} \subseteq F(\gamma)\} = F(\gamma),$$

and similarly,

$$F^*(\gamma) = \{\, u \in U : [u]_R \cap F(\gamma) \neq \emptyset\} = \{\, u \in U : \{u\} \cap F(\gamma) \neq \emptyset\} = F(\gamma).$$

Thus the pair (F_*, F^*, C) reduces to (F, C), which is the original SuperHypersoft Set without approximation. □

3.2 SuperHypersoft Expert Set

The SuperHypersoft Expert Set, a generalization of the Hypersoft Expert Set, is defined as follows.

Definition 3.3 (SuperHypersoft Expert Set)**.** Let Ω be a universe of discourse. Suppose there are n attributes $g_1, \ldots, g_n$, each with a domain G_i. Define the super-attribute domain

$$\mathcal{G} = \mathcal{P}(G_1) \times \mathcal{P}(G_2) \times \cdots \times \mathcal{P}(G_n).$$

Let D be a set of specialists (decision-makers) and C a set of conclusions (e.g., $\{0, 1\}$). Consider

$$T = \mathcal{G} \times D \times C, \quad S \subseteq T.$$

A mapping

$$\Psi : S \longrightarrow \mathcal{P}(\Omega)$$

assigns to each $(\gamma, d, c) \in S$ a subset of Ω. The pair (Ψ, S) is called a *SuperHypersoft Expert Set*.

Theorem 3.4 (Generalization Property of SuperHypersoft Expert Set)**.** *A SuperHypersoft Expert Set generalizes both the Hypersoft Expert Set and the SuperHypersoft Set.*

Proof. We show how each previously known structure is recovered from (Ψ, S):

(i) Generalizing the Hypersoft Expert Set. In the Hypersoft Expert Set, the attribute space is $G_1 \times \cdots \times G_n$. Restrict each $\mathcal{P}(G_i)$ to *singleton* subsets only, so $\mathcal{P}(G_i) \equiv G_i$. Then $\mathcal{G}$ collapses to $G_1 \times \cdots \times G_n$, and the pair (Ψ, S) becomes precisely the Hypersoft Expert Set over $G_1 \times \cdots \times G_n$, with specialists in D and conclusions in C.

(ii) Generalizing the SuperHypersoft Set. If we trivialize the expert dimension ($D = \{*\}$) and the conclusion dimension ($C = \{*\}$), then each element of S is identified simply with $\gamma \in \mathcal{G}$. Hence $\Psi(\gamma, *, *)$ matches a single mapping $F(\gamma)$ where $F : \mathcal{G} \to \mathcal{P}(\Omega)$. This is exactly the SuperHypersoft Set $(F, \mathcal{G})$. □

3.3 Bipolar SuperHypersoft Set

The Bipolar SuperHypersoft Set, a generalization of the Bipolar Hypersoft Set, is defined as follows.

Definition 3.5 (Bipolar SuperHypersoft Set)**.** Let U be a universe, and let $E_1, \ldots, E_n$ be pairwise disjoint sets of parameter values. Define

$$\mathcal{A} = \mathcal{P}(E_1) \times \cdots \times \mathcal{P}(E_n).$$

Let $A \subseteq \mathcal{A}$, and denote by $\neg A$ its complement in $\mathcal{A}$. A *Bipolar SuperHypersoft Set (BSHS-Set)* is a triple (F, G, A) where

$$F : A \to \mathcal{P}(U), \quad G : \neg A \to \mathcal{P}(U),$$

and for every $\alpha \in A$, the following *consistency constraint* holds:

$$F(\alpha) \cap G(\neg\alpha) = \varnothing.$$

Equivalently, we may write

$$(F, G, A) = \left\{ (\alpha, F(\alpha), G(\neg\alpha)) \;\middle|\; \alpha \in A,\ F(\alpha) \cap G(\neg\alpha) = \varnothing \right\}.$$

Theorem 3.6 (Generalization Property of Bipolar SuperHypersoft Set)**.** *A Bipolar SuperHypersoft Set generalizes both the Bipolar Hypersoft Set and the SuperHypersoft Set.*

Proof. We demonstrate how it specializes to each known framework:

(i) Generalizing the Bipolar Hypersoft Set. A Bipolar Hypersoft Set originally has $A \subseteq E_1 \times \cdots \times E_n$. Here, $\mathcal{A} = \mathcal{P}(E_1) \times \cdots \times \mathcal{P}(E_n)$. By restricting each $\mathcal{P}(E_i)$ to singletons only, $\mathcal{A}$ reduces to $E_1 \times \cdots \times E_n$. Hence (F, G, A) satisfies the same constraint $F(\alpha) \cap G(\neg\alpha) = \varnothing$ on that product, which is precisely the definition of a Bipolar Hypersoft Set.

(ii) Generalizing the SuperHypersoft Set. If we remove the bipolar aspect by letting $A = \mathcal{A}$ (i.e. no genuine complement subset) and effectively ignore G, we obtain a single mapping $F : \mathcal{A} \to \mathcal{P}(U)$. This is exactly the usual SuperHypersoft Set. Thus the Bipolar SuperHypersoft Set unifies both concepts into one framework. □

3.4 TreeSoft Rough Set

The TreeSoft Rough Set, a generalization of the Soft Rough Set, is defined as follows.

Definition 3.7 (TreeSoft Rough Set)**.** Let (U, R) be a Pawlak approximation space, and let $(T, \mathrm{Tree}(\mathcal{A}))$ be a TreeSoft Set over U. For each $X \subseteq \mathrm{Tree}(\mathcal{A})$, define

$$T_*(X) = \{u \in U \mid [u]_R \subseteq T(X)\}, \quad T^*(X) = \{u \in U \mid [u]_R \cap T(X) \neq \emptyset\}.$$

Then the *TreeSoft Rough Set* is the triple

$$\big(T_*,\ T^*,\ \mathrm{Tree}(\mathcal{A})\big),$$

where T_* and T^* map each X to subsets of U.

Theorem 3.8. *A TreeSoft Rough Set generalizes both the TreeSoft Set and the Soft Rough Set.*

Proof. (i) Generalizing the TreeSoft Set. Suppose R is the identity relation on U. Then $[u]_R = \{u\}$ for each $u \in U$. Hence, for any $X \subseteq \mathrm{Tree}(\mathcal{A})$,

$$T_*(X) = \{u \in U : [u]_R \subseteq T(X)\} = \{u \in U : \{u\} \subseteq T(X)\} = \{u \in U : u \in T(X)\} = T(X).$$

Similarly,

$$T^*(X) = \{u \in U : [u]_R \cap T(X) \neq \emptyset\} = \{u \in U : \{u\} \cap T(X) \neq \emptyset\} = \{u \in U : u \in T(X)\} = T(X).$$

Thus $T_*(X) = T^*(X) = T(X)$ for all X, so the rough-set structure collapses to the original TreeSoft Set.

(ii) Generalizing the Soft Rough Set. Now let $\text{Tree}(\mathcal{A})$ be a single-level set A, meaning we ignore any sub-attributes. The TreeSoft Set T reduces to a soft set $F : A \to \mathcal{P}(U)$ by identifying $T(\{e\})$ with $F(e)$. In that case,

$$T_*(\{e\}) = \{\, u \in U \mid [u]_R \subseteq T(\{e\})\} = \{\, u \in U \mid [u]_R \subseteq F(e)\},$$

$$T^*(\{e\}) = \{\, u \in U \mid [u]_R \cap T(\{e\}) \neq \emptyset\} = \{\, u \in U \mid [u]_R \cap F(e) \neq \emptyset\},$$

which are precisely the soft lower and soft upper approximations for a Soft Rough Set.

Hence, a TreeSoft Rough Set includes both scenarios as special cases, proving the claim. □

3.5 TreeSoft Expert Set

The TreeSoft Expert Set, a generalization of the Soft Expert Set, is defined as follows.

Definition 3.9 (TreeSoft Expert Set)**.** Let $\text{Tree}(\mathcal{A})$ be as above, X a set of experts, and $O = \{0, 1\}$ the set of opinions. Define

$$\mathcal{D} \;=\; \mathcal{P}\big(\text{Tree}(\mathcal{A})\big) \times X \times O.$$

A *TreeSoft Expert Set* is a pair (Θ, S) where $S \subseteq \mathcal{D}$ and

$$\Theta : S \to \mathcal{P}(U).$$

Each element of S is (Z, x, o) with $Z \subseteq \text{Tree}(\mathcal{A}), x \in X, o \in O$.

Theorem 3.10. *A TreeSoft Expert Set generalizes both the TreeSoft Set and the Soft Expert Set.*

Proof. (i) Generalizing the TreeSoft Set. Take $X = \{*\}$ (only one "expert") and $O = \{*\}$ (only one "opinion"). Then

$$\mathcal{D} = \mathcal{P}(\text{Tree}(\mathcal{A})) \times X \times O \cong \mathcal{P}(\text{Tree}(\mathcal{A})),$$

since $(Z, *, *)$ can be identified with $Z \subseteq \text{Tree}(\mathcal{A})$. Hence $\Theta(Z, *, *) = T(Z)$ for some TreeSoft Set T, recovering the original TreeSoft mapping.

(ii) Generalizing the Soft Expert Set. If we collapse the tree to a single-level set A, we identify $\mathcal{P}(A)$ with A (by restricting to singletons). Then

$$\mathcal{D} = A \times X \times O,$$

and $\Theta(\alpha, x, o)$ precisely matches the usual definition of a Soft Expert Set (F, A) where $F : A \times X \times O \to \mathcal{P}(U)$.

Thus both TreeSoft Set and Soft Expert Set appear as special cases, concluding the proof. □

3.6 Bipolar TreeSoft Set

The Bipolar TreeSoft Set, a generalization of the Bipolar Soft Set, is defined as follows.

Definition 3.11 (Bipolar TreeSoft Set)**.** Let $\text{Tree}(\mathcal{A})$ be a hierarchical attribute tree, and $\mathcal{P}(\text{Tree}(\mathcal{A}))$ its power set. Let $A \subseteq \mathcal{P}(\text{Tree}(\mathcal{A}))$ and $\neg A$ be its complement in $\mathcal{P}(\text{Tree}(\mathcal{A}))$. A *Bipolar TreeSoft Set* is a triple (F, G, A) where

$$F : A \to \mathcal{P}(U), \quad G : \neg A \to \mathcal{P}(U),$$

subject to

$$F(X) \cap G(\neg X) = \varnothing, \quad \forall\, X \in A.$$

Theorem 3.12. *A Bipolar TreeSoft Set generalizes both the TreeSoft Set and the Bipolar Soft Set.*

Proof. (i) Generalizing the TreeSoft Set. If $A = \mathcal{P}(\mathrm{Tree}(\mathcal{A}))$ (i.e., all subsets) and G is trivial (e.g., always returns $\emptyset$), the constraint

$$F(X) \cap G(\neg X) = \emptyset$$

becomes moot. We effectively have just one mapping F on all of $\mathcal{P}(\mathrm{Tree}(\mathcal{A}))$, which is exactly the definition of a TreeSoft Set.

(ii) Generalizing the Bipolar Soft Set. If the tree is replaced by a simple parameter set E, then $\mathcal{P}(\mathrm{Tree}(\mathcal{A}))$ collapses to E. The definitions

$$F : A \to \mathcal{P}(U), \quad G : \neg A \to \mathcal{P}(U), \quad F(e) \cap G(\neg e) = \emptyset,$$

reproduce the usual Bipolar Soft Set exactly.

Hence both structures are special cases, completing the proof. □

4 Future Directions

4.1 Hyperfuzzy Set, Hyperneutrosophic Set, and Hyperplithogenic Set

Looking ahead, this paper explores potential avenues for advancing existing concepts. Within the realm of soft set theory [77,79], various extensions have been thoroughly developed, including Fuzzy Soft Sets [60,123,124], Neutrosophic Soft Sets [3,70,76], and Plithogenic Soft Sets [18]. In the domain of rough set theory, extensions such as Fuzzy Rough Sets [21,30,32] and Neutrosophic Rough Sets [24,140,141] have also gained widespread recognition. Exploring further extensions of these established frameworks remains a promising direction for future work.

Moreover, advanced generalizations of foundational concepts like Fuzzy Sets [128–136], Neutrosophic Sets [37–39, 44–48, 105, 106, 112, 113], and Plithogenic Sets [49, 51, 108, 117] have led to the development of Hyperfuzzy Sets [40, 54, 68, 119], Hyperneutrosophic Sets [28, 40, 42], and Hyperplithogenic Sets [35, 40]. Investigating the feasibility and effectiveness of incorporating these advanced structures into new extensions of soft and rough set theories presents a crucial opportunity for future research endeavors.

4.2 Hyperpolar Fuzzy Set

Another promising direction for future exploration is presented here. In this paper, we have discussed the concept of the Bipolar Soft Set. A closely related concept, known as the Bipolar Fuzzy Set, is well-established in the literature [29,33,57,103,138,139].

It has been demonstrated that the Bipolar Fuzzy Set can be extended to Tripolar Fuzzy Sets [93–96] and further generalized to Multipolar (m-polar) Fuzzy Sets [15,25,53,56,73,97].

An *m-polar fuzzy set* is defined as a mapping that assigns each element of a set X to an m-dimensional vector in $[0,1]^m$, representing membership degrees [25].

We are now considering extending this concept to *hyperpolar fuzzy sets* and *n-superhyperpolar fuzzy sets*. Relevant definitions and related concepts are provided below.

We anticipate that future research will further advance the mathematical structures underlying these concepts and explore their practical applications in various domains.

Definition 4.1. [25] Let X be a non-empty set and $m \in \mathbb{N}$ be a positive integer. An *m-polar fuzzy set* is defined as a mapping $A : X \to [0,1]^m$, where:

- $[0,1]^m$ represents the m-dimensional unit hypercube:

$$[0,1]^m = \{\mathbf{v} = (v_1, v_2, \ldots, v_m) \mid v_i \in [0,1],\ \forall i = 1, 2, \ldots, m\}.$$

- For each $x \in X$, $A(x) = (A_1(x), A_2(x), \ldots, A_m(x)) \in [0,1]^m$, where
$$A_i(x) \in [0,1] \quad \text{for all } i \in \{1, 2, \ldots, m\}.$$

In this context, each component $A_i(x)$ represents the degree of membership of x to the i-th attribute.

Definition 4.2 (Hyperpolar Fuzzy Set)**.** Let X be a non-empty set. Suppose we have a finite collection of positive integers $\delta_1, \delta_2, \ldots, \delta_n \in \mathbb{N}$. Define

$$\Delta = \prod_{k=1}^{n} [0,1]^{\delta_k} = [0,1]^{\delta_1} \times [0,1]^{\delta_2} \times \cdots \times [0,1]^{\delta_n}.$$

A *hyperpolar fuzzy set* on X is a mapping
$$H : X \to \Delta.$$
In other words, for each $x \in X$, $H(x)$ is an n-tuple of vectors, where the k-th vector lies in $[0,1]^{\delta_k}$.

Remark 4.3.

- If $n = 1$ and $\delta_1 = m$, then $H : X \to [0,1]^m$ coincides with the usual notion of an m-polar fuzzy set. Hence every m-polar fuzzy set is a special case of a hyperpolar fuzzy set.
- One may view hyperpolar fuzzy sets as mappings to a product of multiple "fuzzy cubes," each potentially capturing different attribute groups or hierarchical dimensions.

Definition 4.4 (Superhyperpolar Fuzzy Set)**.** Let X be a non-empty set, and let Δ be as in Definition 4.2. A *superhyperpolar fuzzy set* on X is a mapping

$$S : X \to \mathcal{P}(\Delta),$$

where $\mathcal{P}(\Delta)$ is the power set of Δ. Equivalently, for each $x \in X$, $S(x)$ is a *set* of points in Δ (i.e., a set of possible membership vectors).

Remark 4.5. By allowing $S(x)$ to be a *subset* of Δ rather than a single vector in Δ, superhyperpolar fuzzy sets generalize hyperpolar fuzzy sets in much the same way that superhypersoft sets generalize hypersoft sets. In particular, if one identifies each singleton $\{\mathbf{v}\}$ with the vector $\mathbf{v}$ itself, any hyperpolar fuzzy set $H : X \to \Delta$ can be injected into a superhyperpolar fuzzy set $S : X \to \mathcal{P}(\Delta)$ by $S(x) = \{H(x)\}$.

Theorem 4.6.

1. *Every m-polar fuzzy set is a special case of a hyperpolar fuzzy set.*
2. *Every hyperpolar fuzzy set is a special case of a superhyperpolar fuzzy set.*
   In symbols,
   $$\{\textit{m-polar fuzzy sets}\} \subset \{\textit{hyperpolar fuzzy sets}\} \subset \{\textit{superhyperpolar fuzzy sets}\}.$$

Proof.

(1) Let $A : X \to [0,1]^m$ be an m-polar fuzzy set. Set $n = 1$ and $\delta_1 = m$ in Definition 4.2. Then $\Delta = [0,1]^m$, and any mapping $A : X \to [0,1]^m$ is trivially a mapping $A : X \to \Delta$. Hence A is a hyperpolar fuzzy set with just one block of dimension m.

(2) Let $H : X \to \Delta$ be a hyperpolar fuzzy set. Define $S : X \to \mathcal{P}(\Delta)$ by
$$S(x) = \{H(x)\} \subseteq \Delta,$$
which is a singleton for each x. Thus S is a superhyperpolar fuzzy set (Definition 4.4), and H is embedded into S.

Since at least some superhyperpolar fuzzy sets assign more than one membership vector to a given $x \in X$ (i.e., $S(x)$ can be a larger subset of Δ), the inclusion is strict.

□

Funding

This study did not receive any financial or external support from organizations or individuals.

Acknowledgments

We extend our sincere gratitude to everyone who provided insights, inspiration, and assistance throughout this research. We particularly thank our readers for their interest and acknowledge the authors of the cited works for laying the foundation that made our study possible. We also appreciate the support from individuals and institutions that provided the resources and infrastructure needed to produce and share this paper. Finally, we are grateful to all those who supported us in various ways during this project.

Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

References

[1] Tasawar Abbas, Rehan Zafar, Sana Anjum, Ambreen Ayub, and Zamir Hussain. An innovative soft rough dual hesitant fuzzy sets and dual hesitant fuzzy soft rough sets. *VFAST Transactions on Mathematics*, 2023.

[2] Nehmat Ahmed and Osama T. Pirbal. Plithogenic crisp hypersoft topology. *European Journal of Pure and Applied Mathematics*, 2024.

[3] D. Ajay and P. Chellamani. Pythagorean neutrosophic soft sets and their application to decision-making scenario. *Intelligent and Fuzzy Techniques for Emerging Conditions and Digital Transformation*, 2021.

[4] Muhammad Akram, Ghous Ali, and José Carlos R Alcantud. New decision-making hybrid model: intuitionistic fuzzy n-soft rough sets. *Soft Computing*, 23:9853–9868, 2019.

[5] Mohammed Abdullah Al-Hagery, Abdalla I Abdalla Musa, et al. Automated credit card risk assessment using fuzzy parameterized neutrosophic hypersoft expert set. *International Journal of Neutrosophic Science*, (1):93–3, 2025.

[6] Yousef Al-Qudah and Nasruddin Hassan. Bipolar fuzzy soft expert set and its application in decision making. *Int. J. Appl. Decis. Sci.*, 10:175–191, 2017.

[7] Ashraf Al-Quran and Nasruddin Hassan. Neutrosophic vague soft expert set theory. *J. Intell. Fuzzy Syst.*, 30:3691–3702, 2016.

[8] Ashraf Al-Quran and Nasruddin Hassan. The complex neutrosophic soft expert set and its application in decision making. *Journal of Intelligent & Fuzzy Systems*, 34(1):569–582, 2018.

[9] Ashraf Al-Quran, Nasruddin Hassan, and Shawkat Alkhazaleh. Fuzzy parameterized complex neutrosophic soft expert set for decision under uncertainty. *Symmetry*, 11(3):382, 2019.

[10] Tareq M. Al-shami. Bipolar soft sets: Relations between them and ordinary points and their applications. *Complex.*, 2021:6621854:1–6621854:14, 2021.

[11] Faisal Al-Sharqi, Abd Ghafur Ahmad, and Ashraf Al-Quran. Interval-valued neutrosophic soft expert set from real space to complex space. *CMES-Computer Modeling in Engineering & Sciences*, 132(1), 2022.

[12] Faisal Al-Sharqi, Abd Ghafur Ahmad, and Ashraf Al-Quran. Interval-valued neutrosophic soft expert set from real space to complex space. *Computer Modeling in Engineering & Sciences*, 2022.

[13] Faisal Al-Sharqi, Ashraf Al-Quran, et al. Similarity measures on interval-complex neutrosophic soft sets with applications to decision making and medical diagnosis under uncertainty. *Neutrosophic Sets and Systems*, 51:495–515, 2022.

[14] Faisal Al-Sharqi, Ashraf Al-Quran, Abd Ghafur, Said Broumi, and Abd Ghafur Bin Ahmad. Interval-valued complex neutrosophic soft set and its applications in decision-making. 2021.

[15] Bayan Albishry, Sarah O. Alshehri, and Nasr A. Zeyada. Doubt m-polar fuzzy sets based on bck-algebras. *International Journal of Analysis and Applications*, 2023.

[16] Khaleed Alhazaymeh and Nasruddin Hassan. Generalized vague soft expert set theory. 2013.

[17] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *J. Intell. Fuzzy Syst.*, 33:4077–4087, 2017.

[18] Shawkat Alkhazaleh. Plithogenic soft set. *Neutrosophic Sets and Systems*, 33:16, 2020.

[19] Shawkat Alkhazaleh, Abdul Razak Salleh, Nasruddin Hassan, et al. Possibility fuzzy soft set. *Advances in Decision Sciences*, 2011, 2011.

[20] Ali Alqazzaz and Karam M Sallam. Evaluation of sustainable waste valorization using treesoft set with neutrosophic sets. *Neutrosophic Sets and Systems*, 65(1):9, 2024.

[21] Zhan ao Xue, Li ping Zhao, Lin Sun, M. Zhang, and Tianyu Xue. Three-way decision models based on multigranulation support intuitionistic fuzzy rough sets. *Int. J. Approx. Reason.*, 124:147–172, 2020.

[22] Baravan A. Asaad, Sagvan Y. Musa, and Zanyar A. Ameen. Fuzzy bipolar hypersoft sets: A novel approach for decision-making applications. *Mathematical and Computational Applications*, 2024.

[23] Chen Bin. Interval valued generalised fuzzy soft expert set and its application. In *International Conference on Natural Computation, Fuzzy Systems and Knowledge Discovery*, 2019.

[24] Chunxin Bo, Xiaohong Zhang, Songtao Shao, and Florentin Smarandache. Multi-granulation neutrosophic rough sets on a single domain and dual domains with applications. *Symmetry*, 10:296, 2018.

[25] Juanjuan Chen, Shenggang Li, Shengquan Ma, and Xueping Wang. m-polar fuzzy sets: an extension of bipolar fuzzy sets. *The scientific world journal*, 2014(1):416530, 2014.

[26] Orhan Dalkılıç. A decision-making approach to reduce the margin of error of decision makers for bipolar soft set theory. *International Journal of Systems Science*, 53:265 – 274, 2021.

[27] Orhan Dalkılıç and Naime Demirtaş. Decision analysis review on the concept of class for bipolar soft set theory. *Computational and Applied Mathematics*, 41, 2022.

[28] Somen Debnath. *Introduction to Hyper Neutrosophic Set*. Infinite Study, 2020.

[29] O. R. Dehghan. Study of bipolar fuzzy soft hypervector spaces. 2023.

[30] Mustafa Demirci. Genuine sets, various kinds of fuzzy sets and fuzzy rough sets. *Int. J. Uncertain. Fuzziness Knowl. Based Syst.*, 11:467–494, 2003.

[31] NAIME DEMIRTAS, SABIR HUSSAIN, and ORHAN DALKILIC. New approaches of inverse soft rough sets and their applications in a decision making problem. *Journal of applied mathematics & informatics*, 38(3_4):335–349, 2020.

[32] Didier Dubois and Henri Prade. Rough fuzzy sets and fuzzy rough sets. *International Journal of General System*, 17(2-3):191–209, 1990.

[33] Adeel Farooq, Mohammed M. Ali Al-Shamiri, Mohammed M. Khalaf, and Umaira Amjad. Decision-making approach with complex bipolar fuzzy n-soft sets. *Mathematical Problems in Engineering*, 2022.

[34] Feng Feng, Xiaoyan Liu, Violeta Leoreanu-Fotea, and Young Bae Jun. Soft sets and soft rough sets. *Information Sciences*, 181(6):1125–1137, 2011.

[35] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[36] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[37] Takaaki Fujita. Note for neutrosophic incidence and threshold graph. *SciNexuses*, 1:97–125, 2024.

[38] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[39] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[40] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[41] Takaaki Fujita. A comprehensive discussion on fuzzy hypersoft expert, superhypersoft, and indetermsoft graphs. *Neutrosophic Sets and Systems*, 77:241–263, 2025.

[42] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[43] Takaaki Fujita. Short introduction to rough, hyperrough, superhyperrough, treerough, and multirough set. 2025.

[44] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[45] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[46] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[47] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[48] Takaaki Fujita and Florentin Smarandache. Neutrosophic circular-arc graphs and proper circular-arc graphs. *Neutrosophic Sets and Systems*, 78:1–30, 2024.

[49] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[50] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[51] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[52] Oswaldo Edison García Brito, Andrea Sofía Ribadeneira Vacacela, Carmen Hortensia Sánchez Burneo, and Mónica Cecilia Jimbo Galarza. English for specific purposes in the medical sciences to strengthen the professional profile of the higher education medicine student: a knowledge representation using superhypersoft sets. *Neutrosophic Sets and Systems*, 74(1):10, 2024.

[53] Harish Garg, Muhammad Riaz, Muhammad Abdullah Khokhar, and Maryam Saba. Correlation measures for cubic m-polar fuzzy sets with applications. *Mathematical Problems in Engineering*, 2021.

[54] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[55] Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Applied Sciences*, 14(19):8799, 2024.

[56] Purbasha Giri. A study on m-polar fuzzy sets and m-polar fuzzy matrix. *International Journal of Fuzzy Mathematical Archive*, 2021.

[57] Muhammad Gulistan, Naveed Yaqoob, Ahmed Elmoasry, and Jawdat Alebraheem. Complex bipolar fuzzy sets: An application in a transport's company. *J. Intell. Fuzzy Syst.*, 40:3981–3997, 2021.

[58] Nasruddin Hassan and Ashraf Al-Quran. Possibility neutrosophic vague soft expert set for decision under uncertainty. 2017.

[59] Khizar Hayat and Tahir Mahmood. Some applications of bipolar soft set: Characterizations of two isomorphic hemi-rings via bsi-h-ideals. *British Journal of Mathematics & Computer Science*, 13:1–21, 2016.

[60] Khizar Hayat, Raja Aqib Shamim, Hussain Alsalman, Abdu H. Gumaei, Xiaopeng Yang, and Muhammad Azeem Akbar. Group generalized q-rung orthopair fuzzy soft sets: New aggregation operators and their applications. *Mathematical Problems in Engineering*, 2021.

[61] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Hypersoft expert set with application in decision making for recruitment process. 2021.

[62] Muhammad Ihsan, Atiqe Ur Rahman, and Muhammad Haris Saeed. Single valued neutrosophic hypersoft expert set with application in decision making. 2021.

[63] Muhammad Ihsan, Muhammad Haris Saeed, Agaeb Mahal Alanzi, and Hamiden Abd El-Wahed Khalifa. An algorithmic multiple attribute decision-making method for heart problem analysis under neutrosophic hypersoft expert set with fuzzy parameterized degree-based setting. *PeerJ Computer Science*, 9, 2023.

[64] Muhammad Ihsan, Muhammad Haris Saeed, and Atiqe Ur Rahman. An intuitionistic fuzzy hypersoft expert set-based robust decision-support framework for human resource management integrated with modified topsis and correlation coefficient. *Neural Computing and Applications*, pages 1–25, 2023.

[65] Muhammad Ihsan, Muhammad Haris Saeed, Atiqe Ur Rahman, Hüseyin Kamac, Nehad Ali Shah, and Wajaree Weera. An madm-based fuzzy parameterized framework for solar panels evaluation in a fuzzy hypersoft expert set environment. *AIMS Mathematics*, 2022.

[66] Muhammad Ihsan, Muhammad Haris Saeed, Atiqe Ur Rahman, Hamiden A. Wahed Khalifa, and Salwa El-Morsy. An intelligent fuzzy parameterized multi-criteria decision-support system based on intuitionistic fuzzy hypersoft expert set for automobile evaluation. *Advances in Mechanical Engineering*, 14, 2022.

[67] Pablo Andrés Jaramillo Arias, Carlos Javier Torres Miño, Jorge Israel Rivadeneira Cepeda, and Hugo Xavier Cuesta Subia. Analysis of genetic components of late blight resistance (phytophthora infestans) in potatoes (solanum phureja) from cutuglahua, pichincha, using neutrosophic superhypersoft sets. *Neutrosophic Sets and Systems*, 74(1):30, 2024.

[68] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[69] Hüseyin Kamacı. On hybrid structures of hypersoft sets and rough sets. *International Journal of Modern Science and Technology*, 6(4):69–82, 2021.

[70] Faruk Karaaslan. *Neutrosophic soft sets with applications in decision making*. Infinite Study, 2014.

[71] Faruk Karaaslan and Naim Çağman. Bipolar soft rough sets and their applications in decision making. *Afrika Matematika*, 29:823–839, 2018.

[72] Asghar Khan, Muhammad Izhar, and Mohammed M. Khalaf. Generalised multi-fuzzy bipolar soft sets and its application in decision making. *J. Intell. Fuzzy Syst.*, 37:2713–2725, 2019.

[73] Wiyada Kumam, Khalid Naeem, Muhammad Riaz, Muhammad Jabir Khan, and Poom Kumam. Comparison measures for pythagorean $ m $-polar fuzzy sets and their applications to robotics and movie recommender system. *AIMS Mathematics*, 2023.

[74] Zhaowen Li and Tusheng Xie. The relationship among soft sets, soft rough sets and topologies. *Soft Computing*, 18(4):717–728, 2014.

[75] María Eugenia Lucena de Ustáriz, Francisco Javier Ustáriz-Fajardo, Adriana Monserrath Monge Moreno, Adriana Isabel Rodríguez Basantes, Lisbeth Josefina Reales Chacón, Rosa Elisa Cruz Tenempaguay, Verónica Paulina Cáceres Manzano, and Nathaly Kassandra Moscoso Moreno. Antibacterial and antifungal properties of hedyosmum cuatrecazanum occhioni essential oil: A promising natural alternative studied using neutrosophic superhypersoft sets. *Neutrosophic Sets and Systems*, 74(1):3, 2024.

[76] Pabitra Kumar Maji. *Neutrosophic soft set*. Infinite Study, 2013.

[77] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[78] Roshdey Mareay. Soft rough sets based on covering and their applications. *Journal of Mathematics in Industry*, 14:1–11, 2024.

[79] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[80] Sagvan Y. Musa. N-bipolar hypersoft sets: Enhancing decision-making algorithms. *PLOS ONE*, 19, 2024.

[81] Sagvan Y. Musa and Baravan A. Asaad. A novel approach towards parameter reduction based on bipolar hypersoft set and its application to decision-making.

[82] Sagvan Y. Musa and Baravan A. Asaad. Bipolar hypersoft sets. *Mathematics*, 2021.

[83] Sagvan Y Musa and Baravan A Asaad. Topological structures via bipolar hypersoft sets. *Journal of Mathematics*, 2022(1):2896053, 2022.

[84] Edwin Collazos Paucar, Jeri G Ramón Ruffner de Vega, Efrén S Michue Salguedo, Agustina C Torres-Rodríguez, and Patricio A Santiago-Saturnino. Analysis using treesoft set of the strategic development plan for extreme poverty municipalities. *Neutrosophic Sets and Systems*, 69(1):3, 2024.

[85] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[86] Zdzislaw Pawlak. Rough set theory and its applications to data analysis. *Cybernetics & Systems*, 29(7):661–688, 1998.

[87] Zdzis law Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[88] Zdzis law Pawlak. *Rough sets: Theoretical aspects of reasoning about data*, volume 9. Springer Science & Business Media, 2012.

[89] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[90] Zdzislaw Pawlak, Lech Polkowski, and Andrzej Skowron. Rough set theory. *KI*, 15(3):38–39, 2001.

[91] Zdzis law Pawlak and Andrzej Skowron. Rudiments of rough sets. *Information sciences*, 177(1):3–27, 2007.

[92] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[93] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[94] M Murali Krishna Rao. Tripolar fuzzy interior ideals and tripolar fuzzy soft interior ideals over semigroups. *Annals of Fuzzy Mathematics and Informatics*, 20(3):243–256, 2020.

[95] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[96] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, (422019):731743, 2019.

[97] Muhammad Riaz, Muhammad Tahir Hamid, Deeba Afzal, Dragan Pamucar, and Yuming Chu. Multi-criteria decision making in robotic agri-farming with q-rung orthopair m-polar fuzzy sets. *PLoS ONE*, 16, 2021.

[98] Akhil Ranjan Roy and PK Maji. A fuzzy soft set theoretic approach to decision making problems. *Journal of computational and Applied Mathematics*, 203(2):412–418, 2007.

[99] Muhammad Saeed, Muhammad Khubab Siddique, Muhammad Ahsan, Muhammad Rayees Ahmad, and Atiqe Ur Rahman. A novel approach to the rudiments of hypersoft graphs. *Theory and Application of Hypersoft Set, Pons Publication House, Brussel*, pages 203–214, 2021.

[100] Mehmet Şahin, İrfan Deli, and Vakkas Uluçay. *Bipolar Neutrosophic Soft Expert Sets*. Infinite Study.

[101] Mehmet Sahin, Vakkas Ulucay, and Said Broumi. An application of bipolar neutrosophic soft expert set. 2018.

[102] Mehmet Sahin, Vakkas Ulucay, and Said Broumi. Bipolar neutrosophic soft expert set theory. *viXra*, 2018.

[103] Sovan Samanta and Madhumangal Pal. Bipolar fuzzy hypergraphs. *International Journal of Fuzzy Logic Systems*, 2(1):17–28, 2012.

[104] Muhammad Shabir, Muhammad Irfan Ali, and Tanzeela Shaheen. Another approach to soft rough sets. *Knowledge-Based Systems*, 40:72–80, 2013.

[105] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[106] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[107] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[108] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[109] Florentin Smarandache. Foundation of the superhypersoft set and the fuzzy extension superhypersoft set: A new vision. *Neutrosophic Systems with Applications*, 11:48–51, 2023.

[110] Florentin Smarandache. New types of soft sets: Hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set. *International Journal of Neutrosophic Science*, 2023.

[111] Florentin Smarandache. *New types of soft sets "hypersoft set, indetermsoft set, indetermhypersoft set, and treesoft set": an improved version*. Infinite Study, 2023.

[112] Florentin Smarandache. Operators for uncertain over/under/off-sets/-logics/probabilities/-statistics. *Neutrosophic Sets and Systems*, 79:493–500, 2025.

[113] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[114] Florentin Smarandache and Daniela Gifu. The power of soft set extensions with healthcare claims data.

[115] Florentin Smarandache and Daniela Gifu. Soft sets extensions: Innovating healthcare claims analysis. *Available at SSRN 5015972*, 2024.

[116] Florentin Smarandache and Daniela Gifu. Soft sets extensions used in bioinformatics. *Procedia Computer Science*, 246:2185–2193, 2024.

[117] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[118] Florentin Smarandache, A. Saranya, A. Kalavathi, and S. Krishnaprakash. Neutrosophic superhypersoft sets. *Neutrosophic Sets and Systems*, 77:41–53, 2024.

[119] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[120] V. S. Subha and R. Selvakumar. An innovative electric vehicle selection with multi-criteria decision-making approach in indian brands on a neutrosophic hypersoft rough set by using reduct and core. *2024 Fourth International Conference on Advances in Electrical, Computing, Communication and Sustainable Technologies (ICAECT)*, pages 1–7, 2024.

[121] V. S. Subha and R. Selvakumar. A novel approach to plithogenic neutrosophic hypersoft rough set and its application to decision making problem with reference to bakery industry. *Communications in Mathematics and Applications*, 2024.

[122] Gazi University, Orhan Dalkılıç, and Naime Demirtaş. Combination of bipolar soft set and soft expert set with application in decision making. 2021.

[123] Dayu Xu, Xuyao Zhang, and Hailin Feng. Generalized fuzzy soft sets theory-based novel hybrid ensemble credit scoring model. *International Journal of Finance & Economics*, 2018.

[124] Dayu Xu, Xuyao Zhang, Jun-Mei Hu, and Jiahao Chen. A novel ensemble credit scoring model based on extreme learning machine and generalized fuzzy soft sets. *Mathematical Problems in Engineering*, 2020:1–12, 2020.

[125] Yong Yang, Xia Tan, and Congcong Meng. The multi-fuzzy soft set and its application in decision making. *Applied Mathematical Modelling*, 37(7):4915–4923, 2013.

[126] Yiyu Yao and Jilin Yang. Granular rough sets and granular shadowed sets: three-way approximations in pawlak approximation spaces. *International Journal of Approximate Reasoning*, 2021.

[127] Yiyu Yao and Jilin Yang. Granular rough sets and granular shadowed sets: Three-way approximations in pawlak approximation spaces. *Int. J. Approx. Reason.*, 142:231–247, 2022.

[128] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[129] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[130] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[131] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[132] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[133] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[134] Lotfi A Zadeh. Fuzzy sets and information granularity. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 433–448. World Scientific, 1996.

[135] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[136] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

[137] Di Zhang, Pi-Yu Li, and Shuang An. N-soft rough sets and its applications. *Journal of Intelligent & Fuzzy Systems*, 40(1):565–573, 2021.

[138] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[139] Wen-Ran Zhang. Bipolar fuzzy sets. 1997.

[140] Hu Zhao and Hongying Zhang. Some results on multigranulation neutrosophic rough sets on a single domain. *Symmetry*, 10:417, 2018.

[141] Hu Zhao and Hongying Zhang. On hesitant neutrosophic rough set over two universes and its application. *Artificial Intelligence Review*, 53:4387 – 4406, 2019.

[142] Yue Zhao. Enhanced exprom methodology using superhypersoft sets for evaluating the quality of cultural industry development in the digital economy with single-valued neutrosophic sets. *Neutrosophic Sets and Systems*, 79:342–354, 2025.

Chapter 13

Short Survey on the Hierarchical Uncertainty of Fuzzy, Neutrosophic, and Plithogenic Sets

Takaaki Fujita [1] *

[1] Independent Researcher, Shinjuku, Shinjuku-ku, Tokyo, Japan.

Abstract

This paper centers on Plithogenic Sets, a highly adaptable framework that extends concepts such as Fuzzy Sets and Neutrosophic Sets, providing remarkable flexibility for modeling complex relationships. Additionally, advanced extensions like Hyperplithogenic Sets and Superhyperplithogenic Sets have been introduced. This study investigates and consolidates the relationships between these sets and other significant concepts, including Intuitionistic Fuzzy Sets, Vague Sets, Picture Fuzzy Sets, Hesitant Fuzzy Sets, Neutrosophic Sets, Quadripartitioned Neutrosophic Sets, and Pentapartitioned Neutrosophic Sets.

Keywords: Fuzzy set, Neutrosophic set, Hyperstructure, Plithogenic set, Hyperplithogenic set
MSC 2010 classifications: 03E72: Fuzzy set theory, 03B52: Fuzzy logic; logic of vagueness

1 Introduction

1.1 Fuzzy Sets, Neutrosophic Sets, and Plithogenic Sets

Set theory, a fundamental branch of mathematics, provides a robust framework for analyzing collections of elements known as "sets" [48, 98, 109, 126, 236, 237]. This paper focuses on three prominent extensions of classical set theory—Fuzzy Sets [251], Neutrosophic Sets [195], and Plithogenic Sets [202]—and explores their potential generalization into advanced frameworks such as Hyperfuzzy Sets [91], HyperNeutrosophic Sets [71], and Hyperplithogenic Sets [71].

Fuzzy Sets extend classical sets by representing elements with degrees of membership, allowing partial truth values that range continuously between 0 and 1 [251]. Neutrosophic Sets further generalize Fuzzy Sets by introducing three independent parameters: truth, indeterminacy, and falsity, each ranging independently between 0 and 1 [194, 195]. These advancements enable more flexible and nuanced modeling of uncertainty, making them applicable to a wide array of real-world problems.

The core focus of this paper is on Plithogenic Sets, a highly versatile concept that generalizes Fuzzy Sets and Neutrosophic Sets, among others [202, 203]. Plithogenic Sets offer significant flexibility in modeling complex relationships. We provide their formal definition along with illustrative real-world examples.

Definition 1.1. [202, 203] Let S be a universal set, and $P \subseteq S$. A *Plithogenic Set PS* is defined as:

$$PS = (P, v, Pv, pdf, pCF)$$

where:

- v is an attribute.
- Pv is the range of possible values for the attribute v.
- $pdf : P \times Pv \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* [1].
- $pCF : Pv \times Pv \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

[1] Please note that the definition of the Degree of Appurtenance Function may vary across different papers. Some papers define the concept using the power set, while others simplify the definition by avoiding the use of the power set [231]. The author has consistently defined the Classical Plithogenic Set without utilizing the power set.

These functions satisfy the following axioms for all $a, b \in Pv$:

1. *Reflexivity of Contradiction Function*:
$$pCF(a,a) = 0$$
2. *Symmetry of Contradiction Function*:
$$pCF(a,b) = pCF(b,a)$$

Example 1.2 (Plithogenic Set with $s = 3$ and $t = 1$ Corresponding to a Neutrosophic Case)**.** This example illustrates how a Plithogenic Set can mirror the structure of a Neutrosophic Set, with the Degree of Appurtenance Function (pdf) providing three distinct membership values (interpreted as truth, indeterminacy, and falsity) and the Degree of Contradiction Function (pCF) consisting of a single component.

- *Universal Set and Subset:* Let S be a universal set, and let $P = \{\text{Alice}, \text{Bob}, \text{Carol}\} \subseteq S$ be the subset of interest (for instance, students or participants).
- *Attribute:* Define the attribute v as "opinion on a new policy."
- *Range of Possible Values:* Let $Pv = \{\text{support}, \text{neutral}, \text{oppose}\}$ be the set of possible values for the attribute v.
- *Degree of Appurtenance Function (pdf):* Since $s = 3$, the function
$$pdf : P \times Pv \ \rightarrow \ [0,1]^3$$
assigns a triple (T, I, F) to each pair $(x, y) \in P \times Pv$, where:
  - T represents the degree of truth (e.g., agreement or membership),
  - I represents the degree of indeterminacy,
  - F represents the degree of falsity (e.g., disagreement or non-membership).

  For instance, one may define:
$$pdf(\text{Alice}, \text{support}) = (0.7, 0.2, 0.1),$$
$$pdf(\text{Alice}, \text{neutral}) = (0.3, 0.5, 0.2),$$
$$pdf(\text{Alice}, \text{oppose}) = (0.1, 0.3, 0.6),$$
$$pdf(\text{Bob}, \text{support}) = (0.8, 0.1, 0.1),$$
$$pdf(\text{Bob}, \text{neutral}) = (0.2, 0.4, 0.4),$$
$$pdf(\text{Bob}, \text{oppose}) = (0.0, 0.2, 0.8),$$
$$pdf(\text{Carol}, \text{support}) = (0.6, 0.2, 0.2),$$
$$pdf(\text{Carol}, \text{neutral}) = (0.3, 0.5, 0.2),$$
$$pdf(\text{Carol}, \text{oppose}) = (0.2, 0.3, 0.5).$$
  These triples convey, for example, that $pdf(\text{Alice}, \text{support}) = (0.7, 0.2, 0.1)$ indicates Alice is 70% in favor (T), 20% uncertain (I), and 10% against (F) when asked about her support for the policy.
- *Degree of Contradiction Function (pCF):* Since $t = 1$, define
$$pCF : Pv \times Pv \ \rightarrow \ [0,1],$$
returning a single contradiction value for each pair of possible values in Pv. For example,
$$pCF(\text{support}, \text{support}) = 0$$
$$pCF(\text{support}, \text{neutral}) = 0.3,$$
$$pCF(\text{support}, \text{oppose}) = 0.8,$$
$$pCF(\text{neutral}, \text{oppose}) = 0.5.$$
  Here:

- A higher value (e.g., 0.8) between “support” and “oppose” indicates a stronger contradiction.
  - The axioms of Plithogenic Sets require reflexivity, $pCF(a,a) = 0$, and symmetry, $pCF(a,b) = pCF(b,a)$.

- *Neutrosophic Interpretation:* Interpreting (T, I, F) as the degrees of truth, indeterminacy, and falsity, respectively, shows that this Plithogenic Set behaves like a Neutrosophic Set. The single contradiction value pCF quantifies how contradictory any two attribute values are.

- *Summary:* We thus obtain a Plithogenic Set

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF)$$

  with $s = 3$ and $t = 1$, illustrating that the usual Neutrosophic structure (T, I, F) can be embedded within a Plithogenic framework. Each element-preference pair is assigned three membership degrees, and each pair of preferences has a single contradiction measure, all respecting reflexivity and symmetry.

Example 1.3 (Plithogenic Set with $s = 3$ and $t = 3$ in a Marketing Context)**.** Consider a more elaborate situation involving multiple contradiction dimensions.

- *Scenario:* Suppose a small focus group is evaluating a new beverage. Let $P = \{\text{Alice}, \text{Bob}, \text{Carol}\}$. Each participant can have varying degrees of preference: $\{\text{like}, \text{neutral}, \text{dislike}\}$. We also want to capture three types of contradictions, for example: *emotional*, *situational*, and *contextual*.

- *Defining the Plithogenic Set:* We construct

$$PS = (P,\ v,\ Pv,\ pdf,\ pCF),$$

  with $s = 3$ (three membership components) and $t = 3$ (three-dimensional contradiction). Specifically:

  - v is the attribute “preference for the new beverage.”
  - $Pv = \{\text{like}, \text{neutral}, \text{dislike}\}$.
  - $pdf : P \times Pv \to [0,1]^3$. For instance, $\mu(\text{Alice}, \text{like})$ might be $(0.8, 0.1, 0.1)$, indicating 80% “like,” 10% “neutral,” and 10% “dislike.”
  - $pCF : Pv \times Pv \to [0,1]^3$. Each pair of preferences gets a 3D contradiction vector, for example $(0.6, 0.4, 0.2)$. The first component could reflect *emotional conflict*, the second could represent *situational conflict*, and the third could account for *contextual conflict*.

- *Degree of Appurtenance (pdf) Example:* Suppose:

$$pdf(\text{Alice}, \text{like}) = (0.8, 0.1, 0.1),$$
$$pdf(\text{Alice}, \text{neutral}) = (0.2, 0.7, 0.1),$$
$$pdf(\text{Alice}, \text{dislike}) = (0.0, 0.3, 0.7).$$

  Similar assignments can be made for Bob and Carol.

- *Degree of Contradiction (pCF) Example:* Define:

  $pCF(\text{like}, \text{like}) = (0, 0, 0)$, $pCF(\text{like}, \text{dislike}) = (0.6, 0.4, 0.2)$, $pCF(\text{neutral}, \text{dislike}) = (0.3, 0.5, 0.4), \ldots$

  By symmetry, $pCF(a,b) = pCF(b,a)$, and by reflexivity, $pCF(a,a) = (0, 0, 0)$.

- *Interpretation:*

  - Each participant’s preferences are captured by a 3D vector in $[0,1]^3$.
  - Contradictions are split into three dimensions (e.g., emotional, situational, contextual).
  - This approach provides a more refined view of how participants’ preferences may conflict across multiple facets.

- *Summary:* A Plithogenic Set with $s = 3$ and $t = 3$ offers greater nuance than models with fewer contradiction dimensions. By distinguishing three distinct contradiction components, researchers can analyze the complexity of participants’ attitudes and the multifaceted nature of any potential conflicts in real-world studies.

This Plithogenic set can be extended into a HyperPlithogenic set and an n-SuperHyperPlithogenic set [71]. The n-SuperHyperPlithogenic set represents a concept designed to express hierarchical uncertainty. When $n = 1$, it simplifies to a HyperPlithogenic set, and when $n = 0$, it reduces to a Plithogenic set.

1.2 Our Contribution in This Paper

This subsection outlines our contributions in this paper. As previously mentioned, the plithogenic set is particularly valuable due to its flexibility in managing uncertain sets. In this study, we revisit the relationships between s, the number of uncertainty parameters, and t, which represents contradiction. These relationships hold true even when properties such as hyper, superhyper, m-polar, interval-valued, off-condition, and dynamic characteristics are incorporated into the plithogenic set framework.

The relationships between the plithogenic set and other types of sets are detailed in Table 1 [2] [3]. These principles are not limited to sets but also extend to other mathematical structures, including Graphs, Topology, and Algebra.

It should be noted that, while the classification of each set of uncertainties can be mechanically organized as described above, it is equally important to understand the context and significance of the uncertainties being addressed, as well as the parameters under given conditions. The author believes that such understanding is essential for advancing applications and conducting meaningful mathematical research.

2 Preliminaries and Definitions

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. While we aim to present the core ideas, an exhaustive exploration of all terms is beyond the scope of this work. Readers interested in further details are encouraged to consult the cited references for additional insights.

2.1 Fundamentals of Set Theory

This subsection offers a concise overview of the basic principles of set theory. For a more detailed treatment, we recommend established references such as [100, 109, 111].

Definition 2.1 (Set)**.** [109] A *set* is a collection of distinct and clearly defined objects, called *elements*. For any object x, it is possible to determine whether x belongs to a given set. If x is an element of a set A, this is denoted as $x \in A$. Sets are commonly represented using curly braces. For example, the set $A = \{1, 2, 3\}$ contains the elements 1, 2, and 3.

Definition 2.2 (Subset)**.** [109] A set A is said to be a *subset* of another set B, written as $A \subseteq B$, if every element of A is also an element of B. This relationship can be expressed as:

$$A \subseteq B \iff \forall x\, (x \in A \implies x \in B).$$

If $A \subseteq B$ but $A \neq B$, A is referred to as a *proper subset* of B, denoted by $A \subset B$.

Definition 2.3 (Universal Set)**.** [109] The *universal set*, denoted by U, represents the set of all objects under consideration within a specific context. Any set A being analyzed is a subset of U. Formally:

$$A \subseteq U \quad \text{for any set } A.$$

Definition 2.4 (Operation)**.** [109] An *operation* is a function or rule that takes elements of a set S and produces another element within S. Formally, an operation $\circ$ on S is defined as:

$$\circ : S \times S \to S.$$

Examples include addition and multiplication, which are operations on the set of real numbers $\mathbb{R}$.

Definition 2.5 (Binary Operation)**.** [30] A *binary operation* on a set S is a function $* : S \times S \to S$ that combines two elements $a, b \in S$ to produce another element $a * b \in S$. For example, addition and subtraction are binary operations on $\mathbb{R}$.

[2] If the target set type (e.g., Fuzzy or simple Neutrosophic) does not inherently track contradiction as a separate concept, $t = 0$ is generally the simpler approach. Conversely, if the goal is to model or preserve a notion of conflict (as in certain Neutrosophic extensions or uncertain approaches), $t = 1$ is more suitable for capturing such nuances. Although the author has primarily focused on $t = 1$ in their work so far, adopting $t = 0$ could also be a reasonable choice for simplification. While studies such as [55, 56, 58, 61–63, 67, 68, 70, 77, 78, 80, 81, 84, 85, 231] discuss $t = 1$, it should be noted that regardless of the value of t, classical uncertain sets (e.g., Fuzzy sets, Neutrosophic sets) can always be generalized.

[3] As noted in [218], the Turiyam Set should be regarded as a Turiyam Neutrosophic Set. Furthermore, the study emphasizes that the Quadripartitioned Neutrosophic Set may be a more appropriate framework for this concept.

s	t	n	*Type of Plithogenic Set*
1	t	0	Fuzzy Set [251,253–255] N-Set (cf. [119])
1	t	1	Hyperfuzzy Set [91,114]
1	t	n	n-SuperHyperFuzzy Set [71]
2	t	0	Intuitionistic Fuzzy Set [14–21], Vague Set [31,37,101], Pythagorean Fuzzy Set [96] (Paraconsistent Set [33,52,128,243,244])
2	t	1	HyperVague Set [71]
2	t	n	n-SuperHyperVague Set [71]
3	t	0	Neutrosophic Set [194,195], Hesitant Fuzzy Set [232,233], Spherical Fuzzy Set [5,92,94], Picture Fuzzy Set [42,190,222,245], (Three-way Fuzzy Set [102,241]), (Ternary Fuzzy Set [240]) (Faillibilist Set, Dialethist Set, Paradoxist Set [197]) (Pseudoparadoxist Set, Tautological Set [197])
3	t	1	HyperNeutrosophic Set [71]
3	t	n	n-SuperHyperNeutrosophic Set [71]
4	t	0	Quadripartitioned Neutrosophic Set [25,35], Double-Valued Neutrosophic Set [49,120,123,134,246], (Ambiguous Set [187,188,192]), (Turiyam Neutrosophic Set [86,183]) (Fuzzy Quadrigeminal Set [9]) (Four-Valued Set [239])
5	t	0	Pentapartitioned Neutrosophic Set [45]
6	t	0	Hexapartitioned Neutrosophic Set
7	t	0	Heptapartitioned Neutrosophic Set [28,147]
8	t	0	Octapartitioned Neutrosophic Set
9	t	0	Nonapartitioned Neutrosophic Set
p	t	0	MultiFuzzy Set
p+q	t	0	MultiIntuitionistic Fuzzy Set
p+q+r	t	0	MultiNeutrosophic Set
s	t	0	Plithogenic Set [202,203]
s	t	1	HyperPlithogenic Set [71]
s	t	n	SuperhyperPlithogenic Set [71]

Table 1: Examples of n-SuperHyperPlithogenic Sets based on parameters s, t, and n.

2.2 Hyperstructure and Superhyperstructure

This subsection explores the concepts of Hyperstructure and Superhyperstructure, which serve as advanced mathematical frameworks for representing hierarchical systems. A *Hyperstructure* builds upon the notion of a powerset, providing a way to model relationships within sets. Extending this idea, a *Superhyperstructure* incorporates the n-th powerset, enabling the representation of multi-layered, hierarchical systems [57,217,219]. The formal definition of the n-th powerset is provided below.

Definition 2.6 (Base Set)**.** A *base set* is a fundamental set S from which more advanced structures, such as powersets and hyperstructures, are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within the specified domain}\}.$$

All elements in structures such as $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.7 (Powerset)**.** [66, 171] The *powerset* of a set S, denoted $\mathcal{P}(S)$, is the set of all subsets of S, including both the empty set and S itself. Formally, it is defined as:

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Definition 2.8 (n-th Powerset)**.** (cf. [66, 191, 217])

The n-th powerset of a set H, denoted by $P_n(H)$, is defined through an iterative process. Starting with the standard powerset, the construction proceeds as follows:

$$P_1(H) = P(H), \quad P_{n+1}(H) = P(P_n(H)), \quad \text{for } n \geq 1.$$

Similarly, the n-th non-empty powerset, denoted by $P_n^*(H)$, is recursively defined as:

$$P_1^*(H) = P^*(H), \quad P_{n+1}^*(H) = P^*(P_n^*(H)).$$

Here, $P^*(H)$ represents the powerset of H with the empty set excluded.

To provide a formal basis for the concepts of Hyperstructures and Superhyperstructures, we proceed with the following definitions and propositions.

Definition 2.9 (Classical Structure)**.** (cf. [191, 217]) A *Classical Structure* is a mathematical construct defined on a non-empty set H, equipped with one or more *Classical Operations* that adhere to specified *Classical Axioms*. Specifically:

A *Classical Operation* is a function of the form:

$$\#_0 : H^m \to H,$$

where $m \geq 1$ is an integer, and H^m denotes the m-fold Cartesian product of H. Examples include addition and multiplication, as found in common algebraic structures such as groups, rings, and fields.

Definition 2.10 (Hyperstructure)**.** (cf. [66, 191, 217]) A *Hyperstructure* extends the concept of a Classical Structure by operating on the powerset of a base set. Formally, it is defined as:

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where S is the base set, $\mathcal{P}(S)$ is its powerset, and $\circ$ represents an operation defined on subsets within $\mathcal{P}(S)$.

Definition 2.11 (n-Superhyperstructure)**.** (cf. [191, 217]) An *n-Superhyperstructure* generalizes a Hyperstructure by utilizing the n-th powerset of a base set. It is formally represented as:

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

where S is the base set, $\mathcal{P}_n(S)$ is the n-th powerset of S, and $\circ$ is an operation defined on elements of $\mathcal{P}_n(S)$.

Fuzzy Sets [71, 91, 114, 229], Neutrosophic Sets [71], Plithogenic Sets [71], Soft Sets [1, 59, 83, 99, 106, 163, 177, 180, 201, 210], Rough Sets [71, 75], and Vague Sets [71] have all been extended using Hyperstructures and n-SuperHyperstructures.

For example, in the case of Fuzzy Sets, these extensions are known as Hyperfuzzy Sets [54, 91, 114–116, 118, 131, 136, 143, 150, 229] and SuperHyperfuzzy Sets [71]. Similarly, for Neutrosophic Sets, the extensions include HyperNeutrosophic Sets [71] and SuperHyperNeutrosophic Sets [71], while for Plithogenic Sets, they include HyperPlithogenic Sets [71] and SuperHyperPlithogenic Sets [71]. This paper examines these concepts in detail. Readers interested in the underlying principles or further details about these concepts are encouraged to consult relevant literature, such as [71], as needed.

Furthermore, concepts beyond sets are also known as hyperstructures or superhyperstructures, such as hypergraphs [23, 27], superhypergraphs [64, 66, 74, 76, 79, 97, 204, 205], superhyperalgebras [2, 125, 191, 207, 225], superhyperdecision-making [56, 69], Superhyper Z-Number [72], superhyperautomata [84], and superhyperfunctions [211, 216].

2.3 Fuzzy Sets and Their Extensions: Hyperfuzzy and Superhyperfuzzy Sets

This section explores the foundational concept of Fuzzy Sets [251–258] alongside its advanced extensions, Hyperfuzzy Sets [26,91,114,149,229] and Superhyperfuzzy Sets [71]. These frameworks expand the traditional notion of fuzziness into multi-layered structures, allowing for a deeper and more flexible characterization of uncertainty. Detailed definitions are outlined in the following subsections.

Definition 2.12. [251, 256] A *fuzzy set* τ in a non-empty universe Y is a mapping $\tau : Y \to [0, 1]$. A *fuzzy relation* on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a *fuzzy relation on* τ if

$$\delta(y, z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y, z \in Y.$$

Definition 2.13 (HyperFuzzy Set)**.** [26, 91, 114, 149, 229] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1])$ is called a *hyperfuzzy set* over X, where $\tilde{P}([0, 1])$ denotes the family of all non-empty subsets of the interval $[0, 1]$.

Definition 2.14 (n-SuperHyperFuzzy Set)**.** [71, 73] Let X be a non-empty set. The n-SuperHyperFuzzy Set is a recursive generalization of fuzzy sets, hyperfuzzy sets, and superhyperfuzzy sets. It is defined as:

$$\tilde{\mu}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0, 1]),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,
  $$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
  represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0, 1])$ is similarly defined for the interval $[0, 1]$.
- $\tilde{\mu}_n$ assigns to each element $A \in \tilde{\mathcal{P}}_n(X)$ a non-empty subset $\tilde{\mu}_n(A) \subseteq [0, 1]$, representing the degrees of membership associated with A at the n-th level.

2.4 Neutrosophic Sets and HyperNeutrosophic Sets

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, which accounts for situations that are neither entirely true nor entirely false [195]. They have been extensively studied in numerous research works [60, 84, 85, 121, 193, 196, 198, 206, 220, 221, 223, 226]. These sets can be further extended to HyperNeutrosophic Sets and SuperHyperNeutrosophic Sets [71]. The relevant definitions are provided below.

Definition 2.15 (Neutrosophic Set)**.** [195] Let X be a given set. A Neutrosophic Set A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 2.16 (HyperNeutrosophic Set)**.** [71, 73] Let X be a non-empty set. A mapping $\tilde{\mu} : X \to \tilde{P}([0, 1]^3)$ is called a *HyperNeutrosophic Set* over X, where $\tilde{P}([0, 1]^3)$ denotes the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ represents a set of neutrosophic membership degrees, each consisting of truth (T), indeterminacy (I), and falsity (F) components, satisfying:

$$0 \leq T + I + F \leq 3.$$

Definition 2.17 (n-SuperHyperNeutrosophic Set)**.** [71, 73] Let X be a non-empty set. An n-SuperHyperNeutrosophic Set is a recursive generalization of Neutrosophic Sets, HyperNeutrosophic Sets, and SuperHyperNeutrosophic Sets. It is defined as:

$$\tilde{A}_n : \tilde{\mathcal{P}}_n(X) \to \tilde{\mathcal{P}}_n([0, 1]^3),$$

where:

- $\tilde{\mathcal{P}}_1(X) = \tilde{\mathcal{P}}(X)$, and for $k \geq 2$,
$$\tilde{\mathcal{P}}_k(X) = \tilde{\mathcal{P}}(\tilde{\mathcal{P}}_{k-1}(X)),$$
represents the k-th nested family of non-empty subsets of X.
- $\tilde{\mathcal{P}}_n([0,1]^3)$ is similarly defined for the unit cube $[0,1]^3$.
- The mapping $\tilde{A}_n$ assigns to each $A \in \tilde{\mathcal{P}}_n(X)$ a subset $\tilde{A}_n(A) \subseteq [0,1]^3$, representing the degrees of truth (T), indeterminacy (I), and falsity (F) for the n-th level subsets of X.

For each $A \in \tilde{\mathcal{P}}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:
$$0 \leq T + I + F \leq 3,$$
where T, I, and F represent the truth, indeterminacy, and falsity degrees, respectively.

2.5 HyperPlithogenic Set

The HyperPlithogenic Set and SuperHyperPlithogenic Set are extensions of the Plithogenic Set, integrating the concepts of hyperstructures and superhyperstructures. These sets can generalize Fuzzy Sets and Neutrosophic Sets. The formal definition of the HyperPlithogenic Set is provided below.

Definition 2.18 (HyperPlithogenic Set)**.** [71,73] Let X be a non-empty set, and let A be a set of attributes. For each attribute $v \in A$, let Pv be the set of possible values of v. A *HyperPlithogenic Set HPS* over X is defined as:
$$HPS = (P, \{v_i\}_{i=1}^n, \{Pv_i\}_{i=1}^n, \{\tilde{pdf}_i\}_{i=1}^n, pCF)$$
where:

- $P \subseteq X$ is a subset of the universe.
- For each attribute v_i, Pv_i is the set of possible values.
- For each attribute v_i, $\tilde{pdf}_i : P \times Pv_i \to \tilde{P}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P$ and attribute value $a_i \in Pv_i$ a set of membership degrees.
- $pCF : \left(\bigcup_{i=1}^n Pv_i\right) \times \left(\bigcup_{i=1}^n Pv_i\right) \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*.

The HyperPlithogenic Set has the following relationships with other sets.

s	t	*Type of HyperPlithogenic Set*
1	1	Hyperfuzzy Set
2	1	Hypervague Set
3	1	Hyperneutrosophic Set

Table 2: Classification of HyperPlithogenic Sets Based on Parameters s and t.

Next, the definition of the n-SuperHyperPlithogenic Set is provided below. This is an extended concept of the HyperPlithogenic Set.

Definition 2.19 (n-SuperHyperPlithogenic Set)**.** [71,73] Let X be a non-empty set, and let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of attributes, each associated with a set of possible values P_{v_i}. An n-SuperHyperPlithogenic Set ($SHPS_n$) is defined recursively as:
$$SHPS_n = (P_n, V, \{P_{v_i}\}_{i=1}^n, \{\tilde{pdf}_i^{(n)}\}_{i=1}^n, pCF^{(n)}),$$
where:

- $P_1 \subseteq X$, and for $k \geq 2$,
$$P_k = \tilde{\mathcal{P}}(P_{k-1}),$$
represents the k-th nested family of non-empty subsets of P_1.

- For each attribute $v_i \in V$, P_{v_i} is the set of possible values of the attribute v_i.
- For each k-th level subset P_k, $\tilde{pdf}_i^{(n)} : P_n \times P_{v_i} \to \tilde{\mathcal{P}}([0,1]^s)$ is the *Hyper Degree of Appurtenance Function (HDAF)*, assigning to each element $x \in P_n$ and attribute value $a_i \in P_{v_i}$ a subset of $[0,1]^s$.
- $pCF^{(n)} : \bigcup_{i=1}^n P_{v_i} \times \bigcup_{i=1}^n P_{v_i} \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)*, satisfying:
  1. Reflexivity: $pCF^{(n)}(a,a) = 0$ for all $a \in \bigcup_{i=1}^n P_{v_i}$,
  2. Symmetry: $pCF^{(n)}(a,b) = pCF^{(n)}(b,a)$ for all $a, b \in \bigcup_{i=1}^n P_{v_i}$.
- s and t are positive integers representing the dimensions of the membership degrees and contradiction degrees, respectively.

Definition 2.20. Plithogenic Set, HyperPlithogenic Set, and n-SuperHyperPlithogenic Set can be characterized by the following conditions:

- *Bipolar:* The membership degrees are represented by two values: positive membership and negative membership, each taking values in $[0,1]$ (cf. [7, 115, 145, 146, 173, 174]).
- *m-Polar:* The set allows for m-dimensional membership degrees, where each dimension represents a specific perspective or attribute of membership (cf. [4, 6, 89, 90, 175, 176, 182]). For $m = 2$, it corresponds to bipolar; for $m = 3$, it is tripolar (cf. [166–168]).
- *Dynamic:* A concept designed to enable the measurement of changes over time (cf. [32, 129, 135]).
- *q-Rung:* Membership degrees are represented using a q-rung approach, ensuring compatibility with higher levels of uncertainty (cf. [47, 130, 133, 159, 165, 242]). For $q = 2$, it corresponds to Pythagorean (cf. [124, 141, 169, 260, 264]); for $q = 3$, it corresponds to Fermatean (cf. [110, 122, 160, 179]).
- *Complex:* Membership values are extended to the complex plane, allowing for richer representations that include both magnitude and phase (cf. [10, 11, 44, 95, 138, 178, 228, 259]).
- *Multi:* The set accommodates multiple membership degrees simultaneously, often used for multi-criteria decision-making or hierarchical systems (cf. [12, 250]).
- *Linguistic Set:* Membership degrees are represented as linguistic terms (e.g., "low," "medium," "high") rather than numerical values, providing an intuitive interpretation of uncertainty (cf. [87, 130, 265, 266]).
- *Soft Set:* A set characterized by parameterized approximations, where membership depends on a given set of parameters [112, 139].
- *Off Set:* The range of the membership function values is not restricted to $[0,1]$ but can extend to $[\Psi, \Omega]$, where $\Psi < 0$ and $\Omega > 1$ (cf. [142, 153, 199, 200, 224]). The case where $\Psi = 0$ is referred to as an *Over Set* (cf. [208, 209, 238]), while the case where $\Omega = 1$ is called an *Under Set* (cf. [238]).
- *Rough Set:* A set representation based on approximations, defined by a lower and upper bound, which capture incomplete or imprecise information [154–158].

It is known that n-SuperHyperPlithogenic Sets can be transformed as shown in Table 3 [58, 71]. Sets expressible by Classic Plithogenic Sets can be extended using the approach described below.

n	*Type of Plithogenic Set*
0	Plithogenic Set
1	HyperPlithogenic Set
$n > 1$	n-SuperHyperPlithogenic Set

Table 3: Classification of Plithogenic Sets Based on n

3 Result (Revisit)

This section introduces each set and examines the relationships between the sets.

3.1 N-Set

An N-Set is a set defined using a negative-valued function. The formal definition is provided below (cf. [34, 117, 119, 230]).

Definition 3.1. [119] Let X be a non-empty set, and denote by $F(X, [-1,0])$ the collection of all functions from X to $[-1,0]$. Each function in $F(X, [-1,0])$ is called a *negative-valued function* (abbreviated as N-function) on X. An *N-structure* is an ordered pair (X, ρ), where ρ is an N-function on X.

Theorem 3.2. *A Plithogenic set with s=1 can be transformed into an N-Set.*

Proof. This follows directly from the definition. □

3.2 Intuitionistic Fuzzy Set

Intuitionistic Fuzzy Sets extend fuzzy sets by incorporating both membership and non-membership degrees [14, 15, 17, 19, 21]. The definition is provided below.

Definition 3.3. [21] An *intuitionistic fuzzy set* (IFS) A on a universal set E is defined as:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where:

- $\mu_A : E \to [0,1]$ is the membership function, representing the degree to which $x \in E$ belongs to A.
- $\nu_A : E \to [0,1]$ is the non-membership function, representing the degree to which $x \in E$ does not belong to A.
- For every $x \in E$, the following condition holds:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

The *intuitionistic index* $\pi_A(x)$, which represents the degree of hesitation or indeterminacy regarding the membership of x, is given by:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \quad \text{where } \pi_A(x) \geq 0.$$

This hesitation index quantifies the uncertainty in assigning x to the set A.

Theorem 3.4. *A Plithogenic set with s=2 can be transformed into an intuitionistic fuzzy set.*

Proof. This follows directly from the definition. Refer to [70, 82] for additional details if needed. □

3.3 Vague Set: Special Case of Intuitionistic Fuzzy Set

Vague Sets describe elements with truth and false membership functions, defining membership as a range constrained by supporting and opposing evidence [31, 37, 38, 53, 88, 101, 144, 247, 248, 261]. It is known that Vague Sets can be generalized to Intuitionistic Fuzzy Sets [31, 132].

Definition 3.5 (Vague Set)**.** [37, 88] Let U be a universe of discourse, defined as $U = \{u_1, u_2, \ldots, u_n\}$. A *vague set* A in U is characterized by two functions:

$$t_A : U \to [0,1] \quad \text{and} \quad f_A : U \to [0,1],$$

where:

- $t_A(u_i)$ is the *truth-membership function*, providing a lower bound on the membership degree of u_i based on supporting evidence for $u_i \in A$.
- $f_A(u_i)$ is the *false-membership function*, offering a lower bound on the negation of u_i based on evidence against $u_i \in A$.

These functions satisfy the constraint:

$$t_A(u_i) + f_A(u_i) \leq 1, \quad \text{for all } u_i \in U.$$

The degree of membership of u_i in the vague set A is thus constrained within a subinterval of $[0, 1]$ defined by:

$$t_A(u_i) \leq \mu_A(u_i) \leq 1 - f_A(u_i),$$

where $\mu_A(u_i)$ represents the true membership grade of u_i in A. The interval $[t_A(u_i), 1 - f_A(u_i)]$ indicates that, although the exact membership degree may be uncertain, it is bound within this range.

If U is continuous, a vague set A can be represented as:

$$A = \int_U [t_A(u), 1 - f_A(u)]/u.$$

In the case of a discrete universe U, A is expressed as:

$$A = \sum_{i=1}^{n} [t_A(u_i), 1 - f_A(u_i)]/u_i.$$

Theorem 3.6. *A Plithogenic set with s=2 can be transformed into a vague set.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

3.4 Double-valued Neutrosophic Set

A *Double-Valued Neutrosophic Set (DVNS)* is an extension of the Neutrosophic Set, where the indeterminacy degree is divided into two components: the indeterminacy leaning towards truth (I_T) and the indeterminacy leaning towards falsity (I_F) [49, 120, 123, 134, 246].

Definition 3.7. [120] A *Double-Valued Neutrosophic Set* (DVNS) A on a universal set X is defined as:

$$A = \{\langle x, T_A(x), I_T(x), I_F(x), F_A(x)\rangle \mid x \in X\},$$

where:

- $T_A : X \to [0, 1]$ is the *truth membership function*, representing the degree of truth of x.
- $I_T : X \to [0, 1]$ is the *indeterminacy leaning towards truth membership function*, representing the degree of indeterminacy favoring truth.
- $I_F : X \to [0, 1]$ is the *indeterminacy leaning towards falsity membership function*, representing the degree of indeterminacy favoring falsity.
- $F_A : X \to [0, 1]$ is the *falsity membership function*, representing the degree of falsity of x.

For each $x \in X$, the following condition must hold:

$$0 \leq T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4.$$

Theorem 3.8. *A Plithogenic set with s=4 and t=1 can be transformed into a Double-Valued Neutrosophic Set.*

Proof. From the definition of a Plithogenic set, it is characterized by parameters $s = 4$ and $t = 1$, where s represents the dimensions of membership degrees, and t represents the degree of contradiction.

A Double-Valued Neutrosophic Set A is defined with four membership functions: $T_A(x)$, $I_T(x)$, $I_F(x)$, and $F_A(x)$, all mapping to $[0, 1]$, and the constraint

$$0 \leq T_A(x) + I_T(x) + I_F(x) + F_A(x) \leq 4$$

ensures their compatibility within a Plithogenic framework.

By setting the $s = 4$ membership dimensions of the Plithogenic set to $T_A(x)$, $I_T(x)$, $I_F(x)$, and $F_A(x)$, and the $t = 1$ contradiction measure as consistent with the definition of a DVNS (where no explicit contradiction function is utilized), the Plithogenic set directly maps to a Double-Valued Neutrosophic Set.

Thus, the transformation is straightforward and valid. □

3.5 Quadripartitioned Neutrosophic Set

Quadripartitioned Neutrosophic Sets represent elements with four membership degrees: truth, contradiction, ignorance, and falsity [36, 46, 104, 161, 164, 176, 181].

Definition 3.9. [104] A *Quadripartitioned Neutrosophic Set (Q-NS) P* over a universal set Ω is defined as:

$$P = \{(q, T_P(q), C_P(q), G_P(q), F_P(q)) \mid q \in \Omega\},$$

where:

- $T_P : \Omega \to [0, 1]$ is the *truth membership function*, representing the degree of truth of q.
- $C_P : \Omega \to [0, 1]$ is the *contradiction membership function*, representing the degree of contradiction of q.
- $G_P : \Omega \to [0, 1]$ is the *ignorance membership function*, representing the degree of ignorance or indeterminacy of q.
- $F_P : \Omega \to [0, 1]$ is the *falsity membership function*, representing the degree of falsity of q.

For each $q \in \Omega$, the following condition holds:

$$0 \leq T_P(q) + C_P(q) + G_P(q) + F_P(q) \leq 4.$$

Theorem 3.10. *A Plithogenic set with s=4 can be transformed into a Quadripartitioned Neutrosophic Set.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

3.6 Three-way fuzzy set

Three-Way Fuzzy Sets define elements with three distinct membership functions mapping to lattices, capturing nuanced membership degrees [40, 102, 103, 241, 262].

Definition 3.11. [262] Let U be a universe, and let L, M, N be three lattices. A *Three-Way Fuzzy Set (TFS)* is a triple $\langle f, g, h\rangle$, where:

- $f : U \to L$ is the first membership function,
- $g : U \to M$ is the second membership function,
- $h : U \to N$ is the third membership function.

A TFS A is represented mathematically as:

$$A = \{\langle a, f_A(a), g_A(a), h_A(a)\rangle : a \in U, f_A(a) \in L, g_A(a) \in M, h_A(a) \in N\}.$$

Theorem 3.12. *A Plithogenic set with s=3 can be transformed into a Three-way fuzzy Set.*

Proof. This follows directly from the definition. Refer to [71] for additional details if needed. □

3.7 Penta partitioned neutrosophic set

Penta Partitioned Neutrosophic Sets extend neutrosophic sets by incorporating five membership degrees: membership, contradiction, ignorance, unknown, and non-membership [24, 29, 45, 65, 140, 162].

Definition 3.13. [140] Let X be a non-empty set. A Penta-Partitioned Neutrosophic Set A on X is defined as:

$$A = \{(x, \mu_A(x), \sigma_1^A(x), \sigma_2^A(x), \sigma_3^A(x), \gamma_A(x)) : x \in X\},$$

where the functions:

$$\mu_A(x), \sigma_1^A(x), \sigma_2^A(x), \sigma_3^A(x), \gamma_A(x) : X \to [0, 1],$$

represent the following degrees for each $x \in X$:

- $\mu_A(x)$: the degree of membership,
- $\sigma_1^A(x)$: the degree of contradiction,
- $\sigma_2^A(x)$: the degree of ignorance membership,
- $\sigma_3^A(x)$: the degree of unknown membership,
- $\gamma_A(x)$: the degree of non-membership.

These values satisfy the condition:

$$0 \le \mu_A(x) + \sigma_1^A(x) + \sigma_2^A(x) + \sigma_3^A(x) + \gamma_A(x) \le 5, \quad \forall x \in X.$$

Theorem 3.14. *A Plithogenic set with s=5 can be transformed into a Pentapartitioned Neutrosophic Set.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

3.8 Hesitant Fuzzy Set

A Hesitant Fuzzy Set (HFS) allows multiple membership degrees for each element, addressing uncertainty and hesitation in membership assignment [39, 50, 51, 113, 127, 148, 152, 170, 232, 233, 249].

Definition 3.15. [233] A *Hesitant Fuzzy Set (HFS)* on a reference set X is defined as a function:

$$h : X \to P([0, 1]),$$

where $P([0, 1])$ is the power set of the interval $[0, 1]$, and for each $x \in X$, $h(x)$ is a subset of $[0, 1]$, representing the possible membership values of x.

Theorem 3.16. *A Plithogenic set with s=3 can be transformed into a Hesitant Fuzzy Set.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

3.9 Turiyam Neutrosophic Set and Ambiguous Set: Special Cases of the Quadripartitioned/Double-valued Neutrosophic Set

A *Turiyam Neutrosophic Set (TS)* extends the concept of neutrosophic sets by representing elements using four dimensions: truth, indeterminacy, falsity, and liberal membership values [86, 183, 184]. Ambiguous Sets, on the other hand, represent elements with four membership functions: true, false, partially true, and partially false [107, 185, 186, 188, 189, 227]. It is important to note that both Turiyam Neutrosophic Sets and Ambiguous Sets are special cases of Quadripartitioned Neutrosophic Sets or Double-Valued Neutrosophic Sets [192, 212, 214, 215, 227]. And Turiyam Neutrosophic Set is just the Quadruple Neutrosophic Set [218].

Definition 3.17 (Turiyam Neutrosophic Set)**.** [86, 183, 184] Let U be a universal set. A Turiyam Neutrosophic Set T on U is defined as:

$$T = \{\langle x, t_T(x), i_T(x), f_T(x), l_T(x)\rangle : x \in U\},$$

where:

- $t_T(x) : U \to [0, 1]$ represents the truth membership value,
- $i_T(x) : U \to [0, 1]$ represents the indeterminacy membership value,
- $f_T(x) : U \to [0, 1]$ represents the falsity membership value,
- $l_T(x) : U \to [0, 1]$ represents the liberal membership value.

These values satisfy the condition:

$$0 \le t_T(x) + i_T(x) + f_T(x) + l_T(x) \le 4, \quad \forall x \in U.$$

Theorem 3.18. *A Plithogenic set with s=4 and t=1 can be transformed into a Turiyam Neutrosophic Set.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

Definition 3.19. [188, 189] An *Ambiguous Set (AS)* is a mathematical construct defined over a universe U, designed to capture ambiguity in membership degrees using four distinct functions: True Membership Function (TMF), False Membership Function (FMF), Partially True Membership Function (PTMF), and Partially False Membership Function (PFMF). Formally, an AS S for an element $g \in U$ is represented as:

$$S = \{g, \Psi_t(g), \Psi_f(g), \Psi_{ta}(g), \Psi_{fa}(g) \mid g \in U\}$$

Here:

- $\Psi_t(g) : U \to [0, 1]$ represents the *true membership degree*, capturing the evidence supporting g belonging to S.
- $\Psi_f(g) : U \to [0, 1]$ represents the *false membership degree*, capturing the evidence against g belonging to S.
- $\Psi_{ta}(g) : U \to [0, 1]$ represents the *partially true membership degree*, capturing partial or uncertain support for g in S.
- $\Psi_{fa}(g) : U \to [0, 1]$ represents the *partially false membership degree*, capturing partial or uncertain evidence against g in S.

These functions satisfy the constraint:

$$0 \le \Psi_t(g) + \Psi_f(g) + \Psi_{ta}(g) + \Psi_{fa}(g) \le 2$$

Theorem 3.20. *A Plithogenic set with s=4 can be transformed into a Ambiguous Set.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

3.10 Paraconsistent Set

A *Paraconsistent Set* allows both $a \in x$ and $a \notin x$ simultaneously, with truth values $\{0, i, 1\}$ [33,43,52,128,243, 244]. It is known that Paraconsistent Sets can be generalized to Neutrosophic Sets [197,223]. The definition is provided below.

Definition 3.21. [52,243] Let M be a non-empty set. A *Paraconsistent Set* $x \in M$ is characterized as follows:

1. A pair of subsets $([x]_M^+, [x]_M^-)$ where:
$$[x]_M^+ \cup [x]_M^- = M,$$
representing the positive and negative extensions.

2. Membership relations:
$$a \in_M x \iff a \in [x]_M^+, \quad a \notin_M x \iff a \in [x]_M^-.$$

3. A truth function $\epsilon_M : M \times M \to \{0, i, 1\}$:
$$\epsilon_M(a, x) = \begin{cases} 1, & \text{if } a \in_M x \text{ and } a \notin_M x = \emptyset, \\ 0, & \text{if } a \notin_M x \text{ and } a \in_M x = \emptyset, \\ i, & \text{if } a \in_M x \text{ and } a \notin_M x \neq \emptyset. \end{cases}$$

3.11 Four-Valued Set: Special Cases of the Quadripartitioned Neutrosophic Set

A *Four-Valued Set* allows truth values t, f, i, u (*true*, *false*, *inconsistent*, *unknown*) for elements $x \in U \cup \neg U$ [22,239]. The Four-Valued Set is a special case of the Quadripartitioned Neutrosophic Set.

Definition 3.22. [22,239] Let U be a universe, and let $B = \{t, f, i, u\}$ be the set of truth values, where:

- t: true,
- f: false,
- i: inconsistent,
- u: unknown.

A *Four-Valued Set* A is any subset of $U \cup \neg U$, where $\neg U = \{\neg x \mid x \in U\}$ represents the negation of elements in U.

The membership function $\vdash: U \times 2^{U \cup \neg U} \to B$ is defined as:

$$x \vdash A = \begin{cases} t, & \text{if } x \in A \text{ and } \neg x \notin A, \\ i, & \text{if } x \in A \text{ and } \neg x \in A, \\ u, & \text{if } x \notin A \text{ and } \neg x \notin A, \\ f, & \text{if } x \notin A \text{ and } \neg x \in A. \end{cases}$$

The *complement* of a four-valued set A is given by:

$$\neg A = \{\neg x \mid x \vdash A\}.$$

Theorem 3.23. *A Plithogenic set with s=4 and t=1 can be transformed into a Four-Valued Set.*

Proof. This follows directly from the definition. □

3.12 Faillibilist Set, Dialethist Set, Paradoxist Set, Pseudoparadoxist Set, and Tautological Set: Special Case of Neutrosophic Set

In this subsection, we introduce the Faillibilist Set, Dialethist Set, Paradoxist Set, Pseudoparadoxist Set, and Tautological Set. An overview of these concepts is provided below.

The Faillibilist Set, Dialethist Set, Paradoxist Set, Pseudoparadoxist Set, and Tautological Set are special cases of the Neutrosophic Set [196, 197, 223].

1. *Faillibilist Set*: A set where elements have degrees of truth, indeterminacy, and falsity, with indeterminacy strictly greater than zero.
2. *Dialethist Set*: A set allowing contradictions, where some elements are fully true, fully false, and non-indeterminate simultaneously.
3. *Paradoxist Set*: A set with elements exhibiting overindeterminacy, where the indeterminacy value exceeds one for certain elements.
4. *Pseudoparadoxist Set*: A set with overlapping truth and falsity degrees, satisfying $T(x) + F(x) > 1$ and $0 < I(x) < 1$.
5. *Tautological Set*: A set where some elements are underindeterminate, represented by indeterminacy values less than zero for certain elements.

Hereafter, the definitions of each set will be introduced.

Definition 3.24. [196, 197, 223] A *Faillibilist Set A* on a universe X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degree of truth, indeterminacy, and falsity, respectively, for each $x \in X$, and these values satisfy:

$$I_A(x) > 0 \quad \text{(indeterminacy is non-zero)}.$$

Definition 3.25. [196, 197, 223] A *Dialethist Set B* on X allows for contradictions, characterized by:

$$T_B : X \to [0, 1], \quad I_B : X \to [0, 1], \quad F_B : X \to [0, 1],$$

where $T_B(x) = F_B(x) = 1$ and $I_B(x) = 0$ for at least some $x \in X$, allowing an element to simultaneously fully belong to B and its complement.

Definition 3.26. [196, 197, 223] A *Paradoxist Set C* on X is defined by:

$$T_C : X \to [0, 1], \quad I_C : X \to [0, 1], \quad F_C : X \to [0, 1],$$

where $I_C(x) > 1$ for some $x \in X$, indicating overindeterminacy.

Definition 3.27. [196, 197, 223] A *Pseudoparadoxist Set D* on X is defined by:

$$T_D : X \to [0, 1], \quad I_D : X \to [0, 1], \quad F_D : X \to [0, 1],$$

where $0 < I_D(x) < 1$ and $T_D(x) + F_D(x) > 1$ for some $x \in X$, indicating an overlap of truth and falsity.

Definition 3.28. [196, 197, 223] A *Tautological Set E* on X is characterized by:

$$T_E : X \to [0, 1], \quad I_E : X \to [0, 1], \quad F_E : X \to [0, 1],$$

where $I_E(x) < 0$ for some $x \in X$, representing an underindeterminate state.

Theorem 3.29. *Let $PS = (P, v, Pv, pdf, pCF)$ be a Plithogenic Set with $s = 3$ and $t = 1$. Then PS can be transformed into the following specialized sets:*

- *Faillibilist Set,*
- *Dialethist Set,*
- *Paradoxist Set,*
- *Pseudoparadoxist Set,*
- *Tautological Set.*

Proof. To prove the theorem, we analyze the compatibility of the Plithogenic Set structure with the properties of each specialized set.

1. Faillibilist Set: A Faillibilist Set is defined by three membership functions $T(x)$, $I(x)$, and $F(x)$, where $I(x) > 0$ for indeterminacy. With $s = 3$, the Degree of Appurtenance Function pdf provides three values corresponding to $T(x)$, $I(x)$, and $F(x)$. The contradiction function pCF ensures consistency since $t = 1$ implies $pCF(a, a) = 0$. Thus, PS can represent a Faillibilist Set.

2. Dialethist Set: A Dialethist Set allows for elements where $T(x) = F(x) = 1$ and $I(x) = 0$. By assigning $pdf(x) = (1, 0, 1)$, the Plithogenic Set accommodates this property. The symmetric and reflexive nature of pCF under $t = 1$ supports this configuration, making PS a valid Dialethist Set.

3. Paradoxist Set: A Paradoxist Set satisfies $I(x) > 1$ for some x. Although pdf is typically restricted to $[0, 1]^s$, its extension to values greater than 1 is consistent with the Plithogenic framework's flexibility. This enables PS to model overindeterminacy and represent a Paradoxist Set.

4. Pseudoparadoxist Set: A Pseudoparadoxist Set satisfies $0 < I(x) < 1$ and $T(x) + F(x) > 1$ for some x. This can be achieved by assigning $pdf(x)$ values such that $I(x) \in (0, 1)$ and $T(x) + F(x) > 1$, while maintaining pCF consistency for $t = 1$. Hence, PS can serve as a Pseudoparadoxist Set.

5. Tautological Set: A Tautological Set is characterized by $I(x) < 0$ for some x. Extending pdf to include negative values for $I(x)$ remains within the flexible structure of the Plithogenic Set. The reflexivity and symmetry of pCF are unaffected by such extensions, allowing PS to represent a Tautological Set.

By verifying the compatibility of the Plithogenic Set with the defining properties of these specialized sets under $s = 3$ and $t = 1$, the theorem is proven. □

3.13 MultiFuzzy Set, MultiIntuitionistic Fuzzy Set, MultiNeutrosophic Set

The MultiFuzzy Set [12, 263], MultiIntuitionistic Fuzzy Set [213], and MultiNeutrosophic Set [8, 213] are extensions of the Fuzzy Set, Vague Set, Intuitionistic Fuzzy Set, and Neutrosophic Set, respectively. Their definitions are introduced below.

Definition 3.30. (cf. [12, 159, 213, 250, 263]) A *MultiFuzzy Set (MFS)* is a generalization of a fuzzy set where the degree of membership of each element is evaluated by p sources.

Let X be a universe of discourse. A MultiFuzzy Set A on X is defined as:

$$A = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_p(x)) : x \in X\},$$

where $\mu_j(x) : X \to [0, 1]$ for $j = 1, 2, \ldots, p$ represents the degree of membership assigned by the j-th source. Here, $p \geq 2$ ensures the multiplicity of membership.

Definition 3.31. [213] A *MultiIntuitionistic Fuzzy Set (MIFS)* extends the concept of an intuitionistic fuzzy set by allowing p evaluations for membership and q evaluations for non-membership.

Let X be a universe of discourse. A MultiIntuitionistic Fuzzy Set B on X is defined as:

$$B = \{(x, \mu_1(x), \mu_2(x), \ldots, \mu_p(x); \nu_1(x), \nu_2(x), \ldots, \nu_q(x)) : x \in X\},$$

where:

- $\mu_j(x) : X \to [0,1]$ for $j = 1, 2, \ldots, p$, are the degrees of membership,
- $\nu_l(x) : X \to [0,1]$ for $l = 1, 2, \ldots, q$, are the degrees of non-membership,
- $0 \leq \mu_j(x) + \nu_l(x) \leq 1$ for all $j, l, x \in X$.

Here, $p \geq 1$, $q \geq 1$, and $p + q \geq 3$.

Definition 3.32. [8, 213] A *MultiNeutrosophic Set (MNS)* is a generalization of a neutrosophic set where the degrees of truth, indeterminacy, and falsehood are evaluated by p, r, and q sources, respectively.

Let X be a universe of discourse. A MultiNeutrosophic Set C on X is defined as:

$$C = \{(x, T_1(x), T_2(x), \ldots, T_p(x); I_1(x), I_2(x), \ldots, I_r(x); F_1(x), F_2(x), \ldots, F_q(x)) : x \in X\},$$

where:

- $T_j(x) : X \to [0,1]$ for $j = 1, 2, \ldots, p$, are the degrees of truth,
- $I_k(x) : X \to [0,1]$ for $k = 1, 2, \ldots, r$, are the degrees of indeterminacy,
- $F_l(x) : X \to [0,1]$ for $l = 1, 2, \ldots, q$, are the degrees of falsehood,
- $0 \leq \sum_{j=1}^{p} T_j(x) + \sum_{k=1}^{r} I_k(x) + \sum_{l=1}^{q} F_l(x) \leq p + r + q$.

Here, $p, r, q \geq 0$, with at least one of $p, r, q \geq 2$ to ensure multiplicity.

Theorem 3.33. *A Plithogenic Set with $t = 1$ can generalize MultiFuzzy Sets, MultiIntuitionistic Fuzzy Sets, and MultiNeutrosophic Sets.*

Proof. A Plithogenic Set is defined with parameters s (dimensions of membership) and t (dimensions of contradiction). By fixing $t = 1$, the following cases demonstrate its ability to generalize the given sets:

- *MultiFuzzy Set:* For $s = p$ and $t = 1$, the membership functions $\mu_1, \mu_2, \ldots, \mu_p$ of the MultiFuzzy Set correspond to the membership dimensions of the Plithogenic Set. Contradiction is trivial and does not affect the structure.
- *MultiIntuitionistic Fuzzy Set:* For $s = p + q$ and $t = 1$, the membership functions $\mu_1, \mu_2, \ldots, \mu_p$ and non-membership functions $\nu_1, \nu_2, \ldots, \nu_q$ are mapped to the membership dimensions of the Plithogenic Set. The contradiction degree handles potential conflicts between membership and non-membership.
- *MultiNeutrosophic Set:* For $s = p + r + q$ and $t = 1$, the truth functions $T_1, T_2, \ldots, T_p$, indeterminacy functions $I_1, I_2, \ldots, I_r$, and falsehood functions $F_1, F_2, \ldots, F_q$ correspond to the membership dimensions of the Plithogenic Set. The contradiction degree captures inconsistencies among these dimensions.

□

3.14 Picture Fuzzy Sets

A Picture Fuzzy Set (PFS) extends fuzzy sets by defining degrees of positive, neutral, negative, and refusal membership for each element [3, 41, 42, 151, 172, 190, 222, 245].

Definition 3.34 (Picture Fuzzy Set)**.** [42] Let X be a non-empty universe of discourse. A *Picture Fuzzy Set (PFS) A* on X is defined as:

$$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in X\},$$

where:

- $\mu_A(x) : X \to [0,1]$ is the *degree of positive membership* of x in A,
- $\eta_A(x) : X \to [0,1]$ is the *degree of neutral membership* of x in A,
- $\nu_A(x) : X \to [0,1]$ is the *degree of negative membership* of x in A.

These membership functions must satisfy:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X.$$

The *degree of refusal membership* $\rho_A(x)$ is defined as:

$$\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)), \quad \forall x \in X.$$

Theorem 3.35. *Let $PS = (P, v, Pv, pdf, pCF)$ be a Plithogenic Set with $s = 3$. Then PS can be transformed into a Picture Fuzzy Set (PFS) A on a universe X.*

Proof. To prove this theorem, we analyze the structure of PS and show its compatibility with the definition of a Picture Fuzzy Set.

- *Structure of PS:* The Plithogenic Set $PS = (P, v, Pv, pdf, pCF)$ has the following components:
  - P: A subset of a universal set.
  - $pdf : P \times Pv \to [0,1]^s$: Degree of Appurtenance Function (DAF), where $s = 3$.
  - $pCF : Pv \times Pv \to [0,1]^t$: Degree of Contradiction Function (DCF), where $t = 1$.

  For $s = 3$, $pdf(x) = (\mu(x), \eta(x), \nu(x))$ provides three values for each $x \in P$, corresponding to positive, neutral, and negative membership degrees.
- *Compatibility with PFS Definition:* A Picture Fuzzy Set A is defined as:

  $$A = \{\langle x, \mu_A(x), \eta_A(x), \nu_A(x)\rangle : x \in X\},$$

  where $\mu_A(x), \eta_A(x), \nu_A(x)$ satisfy:

  $$\mu_A(x) + \eta_A(x) + \nu_A(x) \leq 1, \quad \forall x \in X.$$

  The degree of refusal membership is defined as:

  $$\rho_A(x) = 1 - (\mu_A(x) + \eta_A(x) + \nu_A(x)).$$

  The Plithogenic Set's $pdf(x)$ directly provides $\mu(x)$, $\eta(x)$, and $\nu(x)$ values, which satisfy:

  $$\mu(x) + \eta(x) + \nu(x) \leq 1,$$

  due to the normalization constraints on pdf. Thus, $pdf(x)$ matches the membership functions of PFS.
- *Role of pCF:* The Degree of Contradiction Function pCF with $t = 1$ ensures that contradictions within Pv are minimized or consistent. This property aligns with the PFS requirement that the sum of membership degrees does not exceed 1, maintaining logical consistency.
- *Conclusion:* By mapping $pdf(x)$ to $(\mu_A(x), \eta_A(x), \nu_A(x))$, the Plithogenic Set PS can represent a Picture Fuzzy Set A. The structural and functional properties of PS satisfy all the requirements of a PFS. Therefore, PS can be transformed into a PFS.

□

3.15 Ternary Fuzzy Set

A Ternary Fuzzy Set extends fuzzy sets by incorporating membership, non-membership, and indeterminacy degrees [240]. This concept can be generalized to a Neutrosophic Set [222].

Definition 3.36. [240] Let X be a non-empty set. A *Ternary Fuzzy Set* $\widetilde{A}$ on X is defined as:

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x), \pi_{\widetilde{A}}(x)) : x \in X\},$$

where the following functions are defined:

- $\mu_{\widetilde{A}} : X \to [0, 1]$: The *degree of membership* of $x \in X$ to $\widetilde{A}$.
- $\nu_{\widetilde{A}} : X \to [0, 1]$: The *degree of non-membership* of $x \in X$ to $\widetilde{A}$.
- $\pi_{\widetilde{A}} : X \to [0, 1]$: The *degree of indeterminacy* of $x \in X$ with respect to $\widetilde{A}$.

These degrees satisfy the following condition for all $x \in X$:

$$0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) + \pi_{\widetilde{A}}(x) \leq 1.$$

Interpretation

- $\mu_{\widetilde{A}}(x)$: Represents the positive support or membership degree.
- $\nu_{\widetilde{A}}(x)$: Represents the opposition or non-membership degree.
- $\pi_{\widetilde{A}}(x)$: Represents the abstention or indeterminacy degree.

Additionally, the following derived measures are defined:

- The *candidate degree* is given by:
$$\rho_{\widetilde{A}}(x) = \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x).$$
- The *non-candidate degree* is:
$$\pi_{\widetilde{A}}(x).$$
- The *remaining indeterminacy degree* is:
$$\tau_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x) - \nu_{\widetilde{A}}(x) - \pi_{\widetilde{A}}(x).$$

Theorem 3.37. *A Plithogenic set with s=3 can be transformed into a Ternary Fuzzy Sets.*

Proof. This follows directly from the definition. □

3.16 Spherical Fuzzy Sets

A Spherical Fuzzy Set (SFS) defines positive, neutral, and negative memberships for elements [13, 93, 94, 105, 108, 137, 234, 235].

Definition 3.38 (Spherical Fuzzy Set)**.** [94] Let $R \neq \emptyset$ be a universe of discourse. A *Spherical Fuzzy Set (SFS)* J on R is defined as:

$$J = \{\langle r, P_J(r), I_J(r), N_J(r)\rangle : r \in R\},$$

where:

- $P_J(r) : R \to [0, 1]$ is the *degree of positive membership*,

- $I_J(r) : R \to [0,1]$ is the *degree of neutral membership*,
- $N_J(r) : R \to [0,1]$ is the *degree of negative membership*.

The membership functions satisfy the constraint:

$$P_J(r)^2 + I_J(r)^2 + N_J(r)^2 \leq 1, \quad \forall r \in R.$$

Theorem 3.39. *A Plithogenic set with s=3 can be transformed into a Spherical Fuzzy Sets.*

Proof. This follows directly from the definition. Refer to [70, 71, 82] for additional details if needed. □

3.17 Heptapartitioned Neutrosophic Set

A Heptapartitioned Neutrosophic Set (HPNS) defines truth, contradiction, unknown, ignorance, and falsity degrees for elements, satisfying specific sum constraints.

Definition 3.40 (Heptapartitioned Neutrosophic Set)**.** [28, 147] Let U be a non-empty universe of discourse. A *Heptapartitioned Neutrosophic Set* (HPNS) A is defined as:

$$A = \{\langle x, T_A(x), M_A(x), C_A(x), U_A(x), I_A(x), K_A(x), F_A(x)\rangle : x \in U\},$$

where:

- $T_A(x) : U \to [0,1]$ is the *truth membership degree* of x in A,
- $M_A(x) : U \to [0,1]$ is the *relative truth degree* of x in A,
- $C_A(x) : U \to [0,1]$ is the *contradiction degree* of x in A,
- $U_A(x) : U \to [0,1]$ is the *unknown membership degree* of x in A,
- $I_A(x) : U \to [0,1]$ is the *ignorance degree* of x in A,
- $K_A(x) : U \to [0,1]$ is the *relative falsity degree* of x in A,
- $F_A(x) : U \to [0,1]$ is the *absolute falsity degree* of x in A.

These membership functions must satisfy the condition:

$$T_A(x) + M_A(x) + C_A(x) + U_A(x) + I_A(x) + K_A(x) + F_A(x) \leq 7, \quad \forall x \in U.$$

Theorem 3.41. *A Plithogenic set with s=7 can be transformed into a Heptapartitioned Neutrosophic Set.*

Proof. This follows directly from the definition. □

4 Conclusion: Relationships Among Various Uncertain Sets

From the discussions above, the relationships outlined in the introductory Table 1 are validated. Furthermore, the concepts of HyperPlithogenic Sets and n-SuperHyperPlithogenic Sets can be understood in accordance with Tables 4 and 5.

Funding

This research did not receive any external funding or financial support.

s	t	*Type of Hyper Set*
1	t	HyperFuzzy Set
2	t	HyperIntuitionistic Fuzzy Set, HyperVague Set, HyperPythagorean Fuzzy Set
3	t	HyperNeutrosophic Set, HyperHesitant Fuzzy Set, HyperSpherical Fuzzy Set, HyperPicture Fuzzy Set
4	t	HyperQuadripartitioned Neutrosophic Set, HyperDouble-Valued Neutrosophic Set
5	t	HyperPentapartitioned Neutrosophic Set
6	t	HyperHexapartitioned Neutrosophic Set
7	t	HyperHeptapartitioned Neutrosophic Set

Table 4: Examples of Hyper Sets based on parameters s and t.

s	t	*Type of n-SuperHyper Set*
1	t	n-SuperHyperFuzzy Set
2	t	n-SuperHyperIntuitionistic Fuzzy Set, n-SuperHyperVague Set, n-SuperHyperPythagorean Fuzzy Set
3	t	n-SuperHyperNeutrosophic Set, n-SuperHyperHesitant Fuzzy Set, n-SuperHyperSpherical Fuzzy Set, n-SuperHyperPicture Fuzzy Set
4	t	n-SuperHyperQuadripartitioned Neutrosophic Set, n-SuperHyperDouble-Valued Neutrosophic Set
5	t	n-SuperHyperPentapartitioned Neutrosophic Set
6	t	n-SuperHyperHexapartitioned Neutrosophic Set
7	t	n-SuperHyperHeptapartitioned Neutrosophic Set

Table 5: Examples of n-SuperHyper Sets based on parameters s and t.

Acknowledgments

We express our sincere gratitude to everyone who offered guidance, encouragement, and support throughout the course of this study. Special thanks are also due to our readers for their interest in this work and to the authors of the cited references, whose foundational insights have significantly contributed to the advancement of our research. Additionally, we deeply appreciate the individuals and organizations who provided the infrastructure necessary for the publication of this paper, as well as all those who have supported us in various ways.

Data Availability

This study is purely theoretical and mathematical, involving no data collection or analysis. Future researchers are encouraged to explore empirical studies or data-driven approaches related to this work.

Ethical Approval

As this research is entirely theoretical, it does not involve human participants or animal subjects, and thus no ethical approval is required.

Conflicts of Interest

The authors declare no conflicts of interest related to this study or its publication.

Disclaimer

This study focuses on theoretical advancements that have not yet been subjected to practical testing or application. Future researchers are encouraged to validate and refine these concepts through empirical studies. While every effort has been made to ensure accuracy and proper citation, inadvertent errors or omissions may occur. Readers are advised to verify referenced materials independently. The views and interpretations expressed in this paper are solely those of the authors and do not necessarily reflect the perspectives of their affiliated institutions.

References

[1] Mujahid Abbas, Ghulam Murtaza, and Florentin Smarandache. *Basic operations on hypersoft sets and hypersoft point*. Infinite Study, 2020.

[2] Fakhry Asad Agusfrianto, Madeleine Al Tahan, and Yudi Mahatma. An introduction to neutrohyperstructures on some chemical reactions. In *NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World*, pages 81–96. IGI Global, 2023.

[3] Dliouah Ahmed and Binxiang Dai. Picture fuzzy rough set and rough picture fuzzy set on two different universes and their applications. *Journal of Mathematics*, 2020.

[4] Muhammad Akram and Arooj Adeel. m-polar fuzzy graphs and m-polar fuzzy line graphs. *Journal of Discrete Mathematical Sciences and Cryptography*, 20:1597 – 1617, 2017.

[5] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[6] Muhammad Akram and Gulfam Shahzadi. Hypergraphs in m-polar fuzzy environment. *Mathematics*, 6(2):28, 2018.

[7] Muhammad Akram, Shumaiza, and Florentin Smarandache. Decision-making with bipolar neutrosophic topsis and bipolar neutrosophic electre-i. *Axioms*, 7:33, 2018.

[8] Tuqa AH Al-Tamimi, Luay AA Al-Swidi, and Ali HM Al-Obaidi. New concepts in partner multineutrosophic topological space. *International Journal of Neutrosophic Science (IJNS)*, 24(3), 2024.

[9] R Alagar and G Albert Asirvatham. Fuzzy quadrigeminal sets: A new approach and application. *Journal of Computational Analysis and Applications (JoCAAA)*, 33(05):268–273, 2024.

[10] Abd Ulzeez M. J. Alkouri, Morad Oqla Massa'deh, and Mabruka Ali. On bipolar complex fuzzy sets and its application. *J. Intell. Fuzzy Syst.*, 39:383–397, 2020.

[11] Sahar Alqaraleh, Abd Ulazeez M. Alkouri, Mourad Oqla Massa'deh, Adeeb G. Talafha, and Anwar Bataihah. Bipolar complex fuzzy soft sets and their application. *Int. J. Fuzzy Syst. Appl.*, 11:1–23, 2022.

[12] B Anitha and P Tharini. A contribution of-neutrosophic multifuzzy subring. *Journal of Algebraic Statistics*, 13(3):3269–3279, 2022.

[13] Shahzaib Ashraf, Saleem Abdullah, Tahir Mahmood, Fazal Ghani, and Tariq Mahmood. Spherical fuzzy sets and their applications in multi-attribute decision making problems. *J. Intell. Fuzzy Syst.*, 36:2829–2844, 2019.

[14] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[15] Krassimir Atanassov and George Gargov. Elements of intuitionistic fuzzy logic. part i. *Fuzzy sets and systems*, 95(1):39–52, 1998.

[16] Krassimir T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20:87–96, 1986.

[17] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[18] Krassimir T. Atanassov. On interval valued intuitionistic fuzzy sets. *Interval-Valued Intuitionistic Fuzzy Sets*, 2019.

[19] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[20] Krassimir T. Atanassov. New topological operator over intuitionistic fuzzy sets. 2023.

[21] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[22] Nuel D Belnap Jr. A useful four-valued logic. In *Modern uses of multiple-valued logic*, pages 5–37. Springer, 1977.

[23] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[24] Pranab Biswas, Surapati Pramanik, and Bibhas Chandra Giri. Single valued bipolar pentapartitioned neutrosophic set and its application in madm strategy. 2022.

[25] Gourangajit Borah and Palash Dutta. Fuzzy risk analysis in crop selection using information measures on quadripartitioned single-valued neutrosophic sets. *Expert Syst. Appl.*, 255:124750, 2024.

[26] Hashem Bordbar, Mohammad Rahim Bordbar, Rajab Ali Borzooei, and Young Bae Jun. N-subalgebras of bck= bci-algebras which are induced from hyperfuzzy structures. *Iranian Journal of Mathematical Sciences and Informatics*, 16(2):179–195, 2021.

[27] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[28] S Broumi and Tomasz Witczak. Heptapartitioned neutrosophic soft set. *International Journal of Neutrosophic Science*, 18(4):270–290, 2022.

[29] Said Broumi, D Ajay, P Chellamani, Lathamaheswari Malayalan, Mohamed Talea, Assia Bakali, Philippe Schweizer, and Saeid Jafari. Interval valued pentapartitioned neutrosophic graphs with an application to mcdm. *Operational Research in Engineering Sciences: Theory and Applications*, 5(3):68–91, 2022.

[30] Richard Hubert Bruck. *A survey of binary systems*, volume 20. Springer, 1971.

[31] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[32] Mingjie Cai, Qingguo Li, and Guangming Lang. Shadowed sets of dynamic fuzzy sets. *Granular Computing*, 2:85 – 94, 2016.

[33] Walter Alexandre Carnielli and Marcelo Esteban Coniglio. Paraconsistent set theory by predicating on consistency. *J. Log. Comput.*, 26:97–116, 2016.

[34] Y Ceven. N-ideals of rings. *International Journal of Algebra*, 6(25):1227–1232, 2012.

[35] Rajashi Chatterjee, Pinaki Majumdar, and Syamal Kumar Samanta. On some similarity measures and entropy on quadripartitioned single valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 30:2475–2485, 2015.

[36] Rajashi Chatterjee, Pinaki Majumdar, and Syamal Kumar Samanta. A multi-criteria group decision making algorithm with quadripartitioned neutrosophic weighted aggregation operators using quadripartitioned neutrosophic numbers in ipqsvnss environment. *Soft Computing*, 24(12):8857–8880, 2020.

[37] Shyi-Ming Chen. Measures of similarity between vague sets. *Fuzzy sets and Systems*, 74(2):217–223, 1995.

[38] Shyi-Ming Chen and Jiann-Mean Tan. Handling multicriteria fuzzy decision-making problems based on vague set theory. *Fuzzy sets and systems*, 67(2):163–172, 1994.

[39] Murat Çolak, İhsan Kaya, Betül Özkan, Ayşenur Budak, and Ali Karaşan. A multi-criteria evaluation model based on hesitant fuzzy sets for blockchain technology in supply chain management. *Journal of Intelligent & Fuzzy Systems*, 38(1):935–946, 2020.

[40] Renato Coppi and Pierpaolo D'Urso. Three-way fuzzy clustering models for lr fuzzy time trajectories. *Computational statistics & data analysis*, 43(2):149–177, 2003.

[41] Bui Cong Cuong. Pythagorean picture fuzzy sets, part 1- basic notions. *Journal of Computer Science and Cybernetics*, 2019.

[42] Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.

[43] Newton C. A. da Costa and Luiz Paulo de Alcantara. On paraconsistent set theories. 1992.

[44] Songsong Dai, Lvqing Bi, and Bo Hu. Distance measures between the interval-valued complex fuzzy sets. *Mathematics*, 2019.

[45] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[46] Somen Debnath. Fuzzy quadripartitioned neutrosophic soft matrix theory and its decision-making approach. *Journal of Computational and Cognitive Engineering*, 1(2):88–93, 2022.

[47] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[48] Keith Devlin. *The joy of sets: fundamentals of contemporary set theory*. Springer Science & Business Media, 1994.

[49] Kanghua Du and Yuming Du. Research on performance evaluation of intangible assets operation and management in sports events with double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:2813–2822, 2023.

[50] Shahzad Faizi etc. Decision making with uncertainty using hesitant fuzzy sets. *International Journal of Fuzzy Systems*, 20:93 – 103, 2017.

[51] Bahram Farhadinia. Correlation for dual hesitant fuzzy sets and dual interval-valued hesitant fuzzy sets. *International Journal of Intelligent Systems*, 29, 2014.

[52] Aldo Figallo-Orellano and Juan Sebastian Slagter. Models for da costa's paraconsistent set theory. 2020.

[53] Kit Fine. Vagueness, truth and logic. 1997.

[54] Takaaki Fujita. General plithogenic soft rough graphs and some related graph classes. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 437.

[55] Takaaki Fujita. Review of plithogenic directed, mixed, bidirected, and pangene offgraph. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 120.

[56] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. Technical report, Center for Open Science, 2024.

[57] Takaaki Fujita. Expanding horizons of plithogenic superhyperstructures: Applications in decision-making, control, and neuro systems. 2024.

[58] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets. 2024. DOI: 10.13140/RG.2.2.12216.87045.

[59] Takaaki Fujita. Note for hypersoft filter and fuzzy hypersoft filter. *Multicriteria Algorithms With Applications*, 5:32–51, 2024.

[60] Takaaki Fujita. Reconsideration of neutrosophic social science and neutrosophic phenomenology with non-classical logic. Technical report, Center for Open Science, 2024.

[61] Takaaki Fujita. A review of fuzzy and neutrosophic offsets: Connections to some set concepts and normalization function. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[62] Takaaki Fujita. Short communication of big plithogenic science and deep plithogenic science. 2024. preprint (researchgate).

[63] Takaaki Fujita. Short note of bunch graph in fuzzy, neutrosophic, and plithogenic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2024.

[64] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[65] Takaaki Fujita. Some graph classes for turiyam neutrosophic directed graphs and pentapartitioned neutrosophic directed graphs. 2024.

[66] Takaaki Fujita. Superhypergraph neural networks and plithogenic graph neural networks: Theoretical foundations. *arXiv preprint arXiv:2412.01176*, 2024.

[67] Takaaki Fujita. Survey of intersection graphs, fuzzy graphs and neutrosophic graphs. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 114, 2024.

[68] Takaaki Fujita. Survey of planar and outerplanar graphs in fuzzy and neutrosophic graphs. *ResearchGate*, July 2024.

[69] Takaaki Fujita. A theoretical exploration of hyperconcepts: Hyperfunctions, hyperrandomness, hyperdecision-making, and beyond (including a survey of hyperstructures). 2024.

[70] Takaaki Fujita. Uncertain labeling graphs and uncertain graph classes (with survey for various uncertain sets). July 2024.

[71] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[72] Takaaki Fujita. Concise note of z-number, hyper z-number, and superhyper z-number. 2025.

[73] Takaaki Fujita. Exploring concepts of hyperfuzzy, hyperneutrosophic, and hyperplithogenic sets ii. *ResearchGate*, 2025.

[74] Takaaki Fujita. Fundamental computational problems and algorithms for superhypergraphs. 2025.

[75] Takaaki Fujita. Short introduction to rough, hyperrough, superhyperrough, treerough, and multirough set. 2025.

[76] Takaaki Fujita and Smarandache Florentin. Some graph parameters for superhypertree-width and neutrosophictree-width. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[77] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[78] Takaaki Fujita and Florentin Smarandache. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[79] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[80] Takaaki Fujita and Florentin Smarandache. Introduction to upside-down logic: Its deep relation to neutrosophic logic and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Third Volume)*. Biblio Publishing, 2024.

[81] Takaaki Fujita and Florentin Smarandache. Mixed graph in fuzzy, neutrosophic, and plithogenic graphs. *Neutrosophic Sets and Systems*, 74:457–479, 2024.

[82] Takaaki Fujita and Florentin Smarandache. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. In *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond (Second Volume)*. Biblio Publishing, 2024.

[83] Takaaki Fujita and Florentin Smarandache. A short note for hypersoft rough graphs. *HyperSoft Set Methods in Engineering*, 3:1–25, 2024.

[84] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[85] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[86] GA Ganati, VNS Rao Repalle, MA Ashebo, and M Amini. Turiyam graphs and its applications. *Information Sciences Letters*, 12(6):2423–2434, 2023.

[87] Harish Garg and Nancy. Algorithms for possibility linguistic single-valued neutrosophic decision-making based on copras and aggregation operators with new information measures. *Measurement*, 2019.

[88] W-L Gau and Daniel J Buehrer. Vague sets. *IEEE transactions on systems, man, and cybernetics*, 23(2):610–614, 1993.

[89] Ganesh Ghorai and Madhumangal Pal. On some operations and density of m-polar fuzzy graphs. *Pacific Science Review A: Natural Science and Engineering*, 17:14–22, 2015.

[90] Ganesh Ghorai and Madhumangal Pal. Faces and dual of m-polar fuzzy planar graphs. *J. Intell. Fuzzy Syst.*, 31:2043–2049, 2016.

[91] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[92] Abhishek Guleria and Rakesh Kumar Bajaj. Eigen spherical fuzzy set and its application in decision making problem. *Scientia Iranica*, 2019.

[93] Fatma Kutlu Gündogdu and Cengiz Kahraman. A novel vikor method using spherical fuzzy sets and its application to warehouse site selection. *J. Intell. Fuzzy Syst.*, 37:1197–1211, 2019.

[94] Fatma Kutlu Gündogdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *J. Intell. Fuzzy Syst.*, 36:337–352, 2019.

[95] Jeonghwan Gwak, Harish Garg, and Naeem Jan. Hybrid integrated decision-making algorithm for clustering analysis based on a bipolar complex fuzzy and soft sets. *Alexandria Engineering Journal*, 2023.

[96] Mostafa Hajiaghaei-Keshteli, Zeynep Cenk, Babek Erdebilli, Yavuz Selim Özdemir, and Fatemeh Gholian-Jouybari. Pythagorean fuzzy topsis method for green supplier selection in the food industry. *Expert Syst. Appl.*, 224:120036, 2023.

[97] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[98] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[99] R. Hema, R. Sudharani, and M. Kavitha. A novel approach on plithogenic interval valued neutrosophic hypersoft sets and its application in decision making. *Indian Journal Of Science And Technology*, 2023.

[100] Doeko Homan. Very basic set theory. 2023.

[101] Dug Hun Hong and Chul Kim. A note on similarity measures between vague sets and between elements. *Information sciences*, 115(1-4):83–96, 1999.

[102] Qingqing Hu and Xiaohong Zhang. Three-way fuzzy sets and their applications (iii). *Axioms*, 12:57, 2023.

[103] Qingqing Hu and Xiaohong Zhang. Three-way fuzzy sets and their applications (iii). *Axioms*, 12(1):57, 2023.

[104] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[105] Scientia Iranica, Abhishek Guleria, and R. K. Bajaj. Eigen spherical fuzzy set and its application to decision-making problem. 2021.

[106] Muhammad Naveed Jafar, Muhammad Haris Saeed, Kainat Muniba Khan, Faten S. Alamri, and Hamiden A. Wahed Khalifa. Distance and similarity measures using max-min operators of neutrosophic hypersoft sets with application in site selection for solid waste management systems. *IEEE Access*, PP:1–1, 2022.

[107] Suchi Jain, Narendra Bisht, Sanjay Oli, Nishant Saxena, Pushpendra Singh Kharayat, and Tarun Pal. Electrical distribution systems classification based on the distance measures of single-valued ambiguous set. In *2024 Asia Pacific Conference on Innovation in Technology (APCIT)*, pages 1–5. IEEE, 2024.

[108] Naeem Jan, Jeonghwan Gwak, Dragan Pamucar, and Hyoungku Kang. An integrated complex t-spherical fuzzy set and soft set model for quantum computing and energy resource planning. *Inf. Sci.*, 661:120101, 2024.

[109] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[110] S. Jeevaraj. Ordering of interval-valued fermatean fuzzy sets and its applications. *Expert Syst. Appl.*, 185:115613, 2021.

[111] Calvin Jongsma. Basic set theory and combinatorics. *Undergraduate Texts in Mathematics*, 2019.

[112] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft graphs: A comprehensive survey. *New Mathematics and Natural Computation*, pages 1–52, 2024.

[113] Juan juan Peng, Jian qiang Wang, and Xiao hui Wu. Novel multi-criteria decision-making approaches based on hesitant fuzzy sets and prospect theory. *Int. J. Inf. Technol. Decis. Mak.*, 15:621–644, 2016.

[114] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[115] Young Bae Jun, Min Su Kang, and Seok Zun Song. Several types of bipolar fuzzy hyper bck-ideals in hyper bck-algebras. 2012.

[116] Young Bae Jun, Seon Jeong Kim, and Seok Zun Song. Hyper permeable values and energetic sets in bck/bci-algebras. 2020.

[117] Young Bae Jun, Mehmet Ali Öztürk, and Eun Hwan Roh. N-structures applied to closed ideals in bch-algebras. *International Journal of Mathematics and Mathematical Sciences*, 2010(1):943565, 2010.

[118] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.

[119] Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 6(6):87, 2018.

[120] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[121] M Kaviyarasu, Muhammad Aslam, Farkhanda Afzal, Maha Mohammed Saeed, Arif Mehmood, and Saeed Gul. The connectivity indices concept of neutrosophic graph and their application of computer network, highway system and transport network flow. *Scientific Reports*, 14(1):4891, 2024.

[122] Mehdi Keshavarz-Ghorabaee, Maghsoud Amiri, Mohammad Hashemi-Tabatabaei, Edmundas Kazimieras Zavadskas, and Art?ras Kaklauskas. A new decision-making approach based on fermatean fuzzy sets and waspas for green construction supplier evaluation. 2020.

[123] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[124] Wiyada Kumam, Khalid Naeem, Muhammad Riaz, Muhammad Jabir Khan, and Poom Kumam. Comparison measures for pythagorean $ m $-polar fuzzy sets and their applications to robotics and movie recommender system. *AIMS Mathematics*, 2023.

[125] M Lathamaheswari, S Sudha, Said Broumi, and Florentin Smarandache. New aspects of neutrosuperhyper algebra with its application. In *NeutroGeometry, NeutroAlgebra, and SuperHyperAlgebra in Today's World*, pages 17–51. IGI Global, 2023.

[126] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[127] Ningna Liao, Guiwu Wei, and Xudong Chen. Todim method based on cumulative prospect theory for multiple attributes group decision making under probabilistic hesitant fuzzy setting. *International Journal of Fuzzy Systems*, 24:322 – 339, 2021.

[128] Thierry Libert. Models for a paraconsistent set theory. *J. Appl. Log.*, 3:15–41, 2005.

[129] Li Liu and Fanzhang Li. A survey on dynamic fuzzy machine learning. *ACM Computing Surveys*, 55:1 – 42, 2022.

[130] Peide Liu, Sumera Naz, Muhammad Akram, and Mamoona Muzammal. Group decision-making analysis based on linguistic q-rung orthopair fuzzy generalized point weighted aggregation operators. *International Journal of Machine Learning and Cybernetics*, 13:883 – 906, 2021.

[131] Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.

[132] An Lu and Wilfred Ng. Vague sets or intuitionistic fuzzy sets for handling vague data: which one is better? In *International conference on conceptual modeling*, pages 401–416. Springer, 2005.

[133] Anam Luqman, Muhammad Akram, and Bijan Davvaz. q-rung orthopair fuzzy directed hypergraphs: A new model with applications. *J. Intell. Fuzzy Syst.*, 37:3777–3794, 2019.

[134] M. M., M. Suneetha, Maria Mikhailova, Sripada Nsvsc Ramesh, and Kollati Vijaya Kumar. Leveraging double-valued neutrosophic set for real-time chronic kidney disease detection and classification. *International Journal of Neutrosophic Science*.

[135] Xuejiao Ma, Yu Jin, and Qingli Dong. A generalized dynamic fuzzy neural network based on singular spectrum analysis optimized by brain storm optimization for short-term wind speed forecasting. *Appl. Soft Comput.*, 54:296–312, 2017.

[136] M MAHARIN. An over view on hyper fuzzy subgroups. *Scholar: National School of Leadership*, 9(1.2), 2020.

[137] Tahir Mahmood, Kifayat Ullah, Qaisar Khan, and Naeem Jan. An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets. *Neural Computing and Applications*, pages 1–13, 2019.

[138] Tahir Mahmood and Ubaid ur Rehman. A novel approach towards bipolar complex fuzzy sets and their applications in generalized similarity measures. *International Journal of Intelligent Systems*, 37:535 – 567, 2021.

[139] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[140] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[141] Wasim Akram Mandal. Bipolar pythagorean fuzzy sets and their application in multi-attribute decision making problems. *Annals of Data Science*, 10:555–587, 2021.

[142] Nivetha Martin, Priya Priya.R, and Florentin Smarandache. Decision making on teachers' adaptation to cybergogy in saturated interval- valued refined neutrosophic overset /underset /offset environment. *International Journal of Neutrosophic Science*, 2020.

[143] Mourad Oqla Massa'deh. On hyper q-fuzzy normal hx subgroup, conjugate and its.

[144] Jaydev Mishra and Sharmistha Ghosh. Uncertain query processing using vague set or fuzzy set: which one is better? *International Journal of Computers Communications & Control*, 9(6):730–740, 2014.

[145] Mai Mohamed and Asmaa Elsayed. A novel multi-criteria decision making approach based on bipolar neutrosophic set for evaluating financial markets in egypt. *Multicriteria Algorithms with Applications*, 2024.

[146] Sagvan Y Musa and Baravan A Asaad. Topological structures via bipolar hypersoft sets. *Journal of Mathematics*, 2022(1):2896053, 2022.

[147] M Myvizhi, Ahmed M Ali, Ahmed Abdelhafeez, and Haitham Rizk Fadlallah. *MADM Strategy Application of Bipolar Single Valued Heptapartitioned Neutrosophic Set*. Infinite Study, 2023.

[148] Sumera Naz and Muhammad Akram. Novel decision-making approach based on hesitant fuzzy sets and graph theory. *Computational and Applied Mathematics*, 38, 2019.

[149] Z Nazari and B Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16(2):173–185, 2018.

[150] Zohreh Nazari and Batool Mosapour. The entropy of hyperfuzzy sets. *Journal of Dynamical Systems and Geometric Theories*, 16:173 – 185, 2018.

[151] etc Nguyen Xuan Thao. Rough picture fuzzy set and picture fuzzy topologies. *Journal of Computer Science and Cybernetics*, 31:245, 2015.

[152] Serif Özlü. Multi-criteria decision making based on vector similarity measures of picture type-2 hesitant fuzzy sets. *Granular Computing*, 8:1505–1531, 2023.

[153] p. geetha and K. Anitha. Single valued neutrosophic soft over / under / offsets. 2016.

[154] Zdzis law Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[155] Zdzis law Pawlak. Rough sets and intelligent data analysis. *Information sciences*, 147(1-4):1–12, 2002.

[156] Zdzis law Pawlak. *Rough sets: Theoretical aspects of reasoning about data*, volume 9. Springer Science & Business Media, 2012.

[157] Zdzislaw Pawlak, Jerzy Grzymala-Busse, Roman Slowinski, and Wojciech Ziarko. Rough sets. *Communications of the ACM*, 38(11):88–95, 1995.

[158] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[159] Mahalakshmi Pethaperumal, Vimala Jeyakumar, Jeevitha Kannan, and Ashma Banu. An algebraic analysis on exploring q-rung orthopair multi-fuzzy sets. *Journal of fuzzy extension and applications*, 4(3):235–245, 2023.

[160] Sarah Qahtan, Hassan A Alsattar, AA Zaidan, Muhammet Deveci, Dragan Pamucar, Dursun Delen, and Witold Pedrycz. Evaluation of agriculture-food 4.0 supply chain approaches using fermatean probabilistic hesitant-fuzzy sets based decision making model. *Applied Soft Computing*, 138:110170, 2023.

[161] R. Radha, A. Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science*, 2021.

[162] Ramakrishnan Radha, A. Stanis, and Arul Mary. Pentapartitioned neutrosophic pythagorean soft set. 2021.

[163] Atiqe Ur Rahman, Muhammad Haris Saeed, and Hamiden Abd El-Wahed Khalifa. Multi-attribute decision-making based on aggregations and similarity measures of neutrosophic hypersoft sets with possibility setting. *Journal of Experimental & Theoretical Artificial Intelligence*, 36:161 – 186, 2022.

[164] M Ramya, Sandesh Murali, and R.Radha. Bipolar quadripartitioned neutrosophic soft set. 2022.

[165] Pratibha Rani and Arunodaya Raj Mishra. Multi-criteria weighted aggregated sum product assessment framework for fuel technology selection using q-rung orthopair fuzzy sets. *Sustainable Production and Consumption*, 24:90–104, 2020.

[166] M Murali Krishna Rao. Tripolar fuzzy interior ideals and tripolar fuzzy soft interior ideals over semigroups. *Annals of Fuzzy Mathematics and Informatics*, 20(3):243–256, 2020.

[167] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[168] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, (422019):731743, 2019.

[169] Muhammad Riaz and Masooma Raza Hashmi. Soft rough pythagorean m-polar fuzzy sets and pythagorean m-polar fuzzy soft rough sets with application to decision-making. *Computational and Applied Mathematics*, 39, 2019.

[170] Rosa M Rodríguez, Luis Martínez, Vicenç Torra, ZS Xu, and Francisco Herrera. Hesitant fuzzy sets: state of the art and future directions. *International journal of intelligent systems*, 29(6):495–524, 2014.

[171] Judith Roitman. *Introduction to modern set theory*, volume 8. John Wiley & Sons, 1990.

[172] Rekha Sahu, Satya Ranjan Dash, and Sujit Das. Career selection of students using hybridized distance measure based on picture fuzzy set and rough set theory. 2021.

[173] Hanan H Sakr, Abdisalam Hassan Muse, and Ramy Aldallal. A generalized decision-making technique based on bipolar-valued multivague soft sets. *Journal of Function Spaces*, 2022(1):9453172, 2022.

[174] Sovan Samanta and Madhumangal Pal. Bipolar fuzzy hypergraphs. *International Journal of Fuzzy Logic Systems*, 2(1):17–28, 2012.

[175] Musavarah Sarwar and Muhammad Akram. New applications of m-polar fuzzy matroids. *Symmetry*, 9:319, 2017.

[176] S Satham Hussain, Durga Nagarajan, Hossein Rashmanlou, and Farshid Mofidnakhaei. Novel supply chain decision making model under m-polar quadripartitioned neutrosophic environment. *Journal of Applied Mathematics and Computing*, pages 1–26, 2024.

[177] P Sathya, Nivetha Martin, and Florentine Smarandache. Plithogenic forest hypersoft sets in plithogenic contradiction based multi-criteria decision making. *Neutrosophic Sets and Systems*, 73:668–693, 2024.

[178] Ganeshsree Selvachandran, Madhumangal Pal, Tahani A. Abdusalam Alhawari, and Abdul Razak Salleh. Interval-valued complex fuzzy sets and its application to the malaysian economy. *Int. J. Fuzzy Syst. Appl.*, 7:22–31, 2018.

[179] Tapan Senapati and Ronald R. Yager. Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, 11:663 – 674, 2019.

[180] Francina Shalini. Trigonometric similarity measures of pythagorean neutrosophic hypersoft sets. *Neutrosophic Systems with Applications*, 2023.

[181] Xiaolong Shi, Saeed Kosari, Hossein Rashmanlou, Said Broumi, and S Satham Hussain. Properties of interval-valued quadripartitioned neutrosophic graphs with real-life application. *Journal of Intelligent & Fuzzy Systems*, 44(5):7683–7697, 2023.

[182] Prem Kumar Singh. M-polar fuzzy graph representation of concept lattice. *Eng. Appl. Artif. Intell.*, 67:52–62, 2018.

[183] Prem Kumar Singh. Turiyam set a fourth dimension data representation. *Journal of Applied Mathematics and Physics*, 9(7):1821–1828, 2021.

[184] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[185] Pritpal Singh. A general model of ambiguous sets to a single-valued ambiguous numbers with aggregation operators. *Decision Analytics Journal*, 8:100260, 2023.

[186] Pritpal Singh. From ambiguous sets to single-valued ambiguous complex numbers: Applications in mandelbrot set generation and vector directions. *Modern Physics Letters B*, page 2450509, 2024.

[187] Pritpal Singh and Surya Sekhar Bose. Ambiguous d-means fusion clustering algorithm based on ambiguous set theory: Special application in clustering of ct scan images of covid-19. *Knowledge-Based Systems*, 231:107432, 2021.

[188] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[189] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[190] Pushpinder Singh. Correlation coefficients for picture fuzzy sets. *J. Intell. Fuzzy Syst.*, 28:591–604, 2015.

[191] F. Smarandache. Introduction to superhyperalgebra and neutrosophic superhyperalgebra. *Journal of Algebraic Hyperstructures and Logical Algebras*, 2022.

[192] Florentin Smarandache. Ambiguous set (as) is a particular case of the quadripartitioned neutrosophic set (qns). *nidus idearum*, page 16.

[193] Florentin Smarandache. Neutrosophic overset, neutrosophic underset, and neutrosophic offset. similarly for neutrosophic over-/under-/offlogic, probability, and statisticsneutrosophic, pons editions brussels, 170 pages book, 2016.

[194] Florentin Smarandache. Neutrosophy: neutrosophic probability, set, and logic: analytic synthesis & synthetic analysis. 1998.

[195] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[196] Florentin Smarandache. Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, 24(3):287, 2005.

[197] Florentin Smarandache. Neutrosophic physics: More problems, more solutions. 2010.

[198] Florentin Smarandache. Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, 1(1):107–116, 2010.

[199] Florentin Smarandache. Degrees of membership¿ 1 and¡ 0 of the elements with respect to a neutrosophic offset. *Neutrosophic Sets and Systems*, 12:3–8, 2016.

[200] Florentin Smarandache. Neutrosophic overset. *Neutrosophic Underset, and Neutrosophic Offset: Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics, Pons Editions Brussels*, 2016.

[201] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[202] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[203] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[204] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[205] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[206] Florentin Smarandache. *NeutroGeometry & AntiGeometry are alternatives and generalizations of the Non-Euclidean Geometries (revisited)*, volume 5. Infinite Study, 2021.

[207] Florentin Smarandache. History of superhyperalgebra and neutrosophic superhyperalgebra (revisited again). *Neutrosophic Algebraic Structures and Their Applications*, page 10, 2022.

[208] Florentin Smarandache. Interval-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers. Volume IX: On Neutrosophic Theory and Its Applications in Algebra*, page 117, 2022.

[209] Florentin Smarandache. Operators on single-valued neutrosophic oversets, neutrosophic undersets, and neutrosophic offsets. *Collected Papers*, 9:112, 2022.

[210] Florentin Smarandache. *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[211] Florentin Smarandache. *The SuperHyperFunction and the Neutrosophic SuperHyperFunction (revisited again)*, volume 3. Infinite Study, 2022.

[212] Florentin Smarandache. Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general. *Neutrosophic Sets & Systems*, 58, 2023.

[213] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[214] Florentin Smarandache. *Nidus Idearum. Scilogs, XI: in-turns and out-turns*. Infinite Study, 2023.

[215] Florentin Smarandache. *Nidus Idearum. Scilogs, XII: seed & heed*. Infinite Study, 2023.

[216] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[217] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[218] Florentin Smarandache. *Nidus Idearum. Scilogs, XIII: Structure/NeutroStructure/AntiStructure*. Infinite Study, 2024.

[219] Florentin Smarandache. Superhyperstructure & neutrosophic superhyperstructure, 2024. Accessed: 2024-12-01.

[220] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group. *Neural Computing and Applications*, 29(7):595–601, 2018.

[221] Florentin Smarandache and Mumtaz Ali. Neutrosophic triplet group (revisited). *Neutrosophic sets and Systems*, 26(1):2, 2019.

[222] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[223] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[224] Florentin Smarandache, Miguel A. Quiroz-Martínez, Jesús Estupiñán, Noel Batista Hernández, and Maikel Yelandi Leyva Vazquez. Application of neutrosophic offsets for digital image processing. 2020.

[225] Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay, and Abdullah Kargın. *Neutrosophic SuperHyperAlgebra and New Types of Topologies*. Infinite Study, 2023.

[226] Florentin Smarandache and AA Salama. Neutrosophic crisp set theory. 2015.

[227] Florentin Smarandache and F Smarandache. Ambiguous set brings nothing new. *nidus idearum*, page 24.

[228] Haifeng Song, Lvqing Bi, Bo Hu, Yingying Xu, and Songsong Dai. New distance measures between the interval-valued complex fuzzy sets with applications to decision-making. *Mathematical Problems in Engineering*, 2021.

[229] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[230] M Songsaeng and A Iampan. N-fuzzy up-algebras and its level subsets. *Journal of Algebra and Related Topics*, 6(1):1–24, 2018.

[231] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[232] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[233] Vicenç Torra and Yasuo Narukawa. On hesitant fuzzy sets and decision. In *2009 IEEE international conference on fuzzy systems*, pages 1378–1382. IEEE, 2009.

[234] Kifayat Ullah, Harish Garg, Tahir Mahmood, Naeem Jan, and Zeeshan Ali. Correlation coefficients for t-spherical fuzzy sets and their applications in clustering and multi-attribute decision making. *Soft Computing*, 24:1647 – 1659, 2019.

[235] Kifayat Ullah, Tahir Mahmood, and Naeem Jan. Similarity measures for t-spherical fuzzy sets with applications in pattern recognition. *Symmetry*, 10:193, 2018.

[236] Robert L Vaught. *Set theory: an introduction*. Springer Science & Business Media, 2001.

[237] Nikolai Konstantinovich Vereshchagin and Alexander Shen. *Basic set theory*. Number 17. American Mathematical Soc., 2002.

[238] Dhatchinamoorthy Vinoth and Devarasan Ezhilmaran. An analysis of global and adaptive thresholding for biometric images based on neutrosophic overset and underset approaches. *Symmetry*, 15:1102, 2023.

[239] Aida Vitória, Andrzej Sza las, and Jan Ma luszyński. Four-valued extension of rough sets. In *Rough Sets and Knowledge Technology: Third International Conference, RSKT 2008, Chengdu, China, May 17-19, 2008. Proceedings 3*, pages 106–114. Springer, 2008.

[240] Chao Wang, Minghu Ha, and Xiaowei Liu. A mathematical model of ternary fuzzy set for voting. *Journal of Intelligent & Fuzzy Systems*, 29(6):2381–2386, 2015.

[241] Jingqian Wang, Xiaohong Zhang, and Qingqing Hu. Three-way fuzzy sets and their applications (ii). *Axioms*, 11(10):532, 2022.

[242] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[243] Zach Weber. Transfinite numbers in paraconsistent set theory. *The Review of Symbolic Logic*, 3:71 – 92, 2010.

[244] Zach Weber. Transfinite cardinals in paraconsistent set theory. *The Review of Symbolic Logic*, 5:269 – 293, 2012.

[245] Guiwu Wei and Hui Gao. The generalized dice similarity measures for picture fuzzy sets and their applications. *Informatica*, 29:107–124, 2018.

[246] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[247] Timothy Williamson. *Vagueness*. Routledge, 2002.

[248] Rong Xing, Rui Cheng, Jiwen Huang, Qing Li, and Jingmei Zhao. Learning to understand the vague graph for stock prediction with momentum spillovers. *IEEE Transactions on Knowledge and Data Engineering*, 2023.

[249] Zeshui Xu. Hesitant fuzzy sets theory. In *Studies in Fuzziness and Soft Computing*, 2014.

[250] Jun Ye, Shigui Du, and Rui Yong. Multifuzzy cubic sets and their correlation coefficients for multicriteria group decision-making. *Mathematical Problems in Engineering*, 2021(1):5520335, 2021.

[251] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[252] Lotfi A Zadeh. Biological application of the theory of fuzzy sets and systems. In *The Proceedings of an International Symposium on Biocybernetics of the Central Nervous System*, pages 199–206. Little, Brown and Comp. London, 1969.

[253] Lotfi A Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges. 1972.

[254] Lotfi A Zadeh. Fuzzy sets and their application to pattern classification and clustering analysis. In *Classification and clustering*, pages 251–299. Elsevier, 1977.

[255] Lotfi A Zadeh. Fuzzy sets versus probability. *Proceedings of the IEEE*, 68(3):421–421, 1980.

[256] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[257] Lotfi A Zadeh. A note on prototype theory and fuzzy sets. In *Fuzzy sets, fuzzy logic, and fuzzy systems: Selected papers by Lotfi A Zadeh*, pages 587–593. World Scientific, 1996.

[258] Lotfi Asker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1(1):3–28, 1978.

[259] Muhammad Zeeshan, Madad Khan, and Sohail Iqbal. Distance function of complex fuzzy soft sets with application in signals. *Computational and Applied Mathematics*, 41, 2022.

[260] Chao Zhang, Juanjuan Ding, Jianming Zhan, and Deyu Li. Incomplete three-way multi-attribute group decision making based on adjustable multigranulation pythagorean fuzzy probabilistic rough sets. *Int. J. Approx. Reason.*, 147:40–59, 2022.

[261] Qinghua Zhang, Jin Wang, Guoyin Wang, and Hong Yu. The approximation set of a vague set in rough approximation space. *Information Sciences*, 300:1–19, 2015.

[262] Xue Rong Zhao and Yiyu Yao. Three-way fuzzy partitions defined by shadowed sets. *Information Sciences*, 497:23–37, 2019.

[263] Xiaoqiang Zhou, Chunyong Wang, and Zichuang Huang. Interval-valued multi-fuzzy soft set and its application in decision making. *Int. J. Comput. Sci. Eng. Technol*, 9:48–54, 2019.

[264] Yang Zhou, Yuan Xu, Wuhuan Xu, Jun Wang, and Guangming Yang. A novel multiple attribute group decision-making approach based on interval-valued pythagorean fuzzy linguistic sets. *IEEE Access*, 8:176797–176817, 2020.

[265] Li Zou, Hongmei Lin, Xiaoying Song, Kaihua Feng, and Xin Liu. Rule extraction based on linguistic-valued intuitionistic fuzzy layered concept lattice. *Int. J. Approx. Reason.*, 133:1–16, 2021.

[266] Li Zou, Xin Liu, Zheng Pei, and Degen Huang. Implication operators on the set of v-irreducible element in the linguistic truth-valued intuitionistic fuzzy lattice. *International Journal of Machine Learning and Cybernetics*, 4:365 – 372, 2012.

The series of *Collected Papers* on **Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond** aims to explore the ongoing evolution of uncertain combinatorics through innovative methodologies such as graphization, hyperization, and uncertainization. These approaches integrate and extend core concepts from fuzzy, neutrosophic, soft, and rough set theories, providing robust frameworks to model and analyze the inherent complexity of real-world uncertainties.

This volume specifically delves into the concept of the **HyperUncertain Set**, building on the foundational advancements introduced in previous volumes.

The book is structured into thirteen chapters, each contributing unique perspectives and advancements in the realm of HyperUncertain Sets and their related frameworks:

1. **Advancing Traditional Set Theory with Hyperfuzzy, Hyperneutrosophic, and Hyperplithogenic Sets**: The opening chapter explores how classical set theory evolves to meet the demands of real-world complexity by incorporating hyperstructures and superhyperstructures. It introduces and expands the frameworks of Hyperfuzzy, Hyperneutrosophic, and Hyperplithogenic Sets, offering applications across diverse fields such as topology, decision-making, and language theory.
2. **Applications and Mathematical Properties of Hyperneutrosophic and SuperHyperneutrosophic Sets**: This chapter delves into advanced applications of Hyperneutrosophic Sets, focusing on their integration into Neutrosophic Logic, Cognitive Maps, Graph Neural Networks, and Triplet Groups, alongside an analysis of their mathematical properties.
3. **New Extensions of Hyperneutrosophic Sets – Bipolar, Pythagorean, Double-Valued, and Interval-Valued Sets**: Here, the text explores extensions like Bipolar and Interval-Valued Neutrosophic Sets, emphasizing their applications in addressing increasingly intricate scenarios.
4. **Hyperneutrosophic Extensions of Complex, Single-Valued Triangular, Fermatean, and Linguistic Sets**: Recent advancements in hyperextensions of Neutrosophic Sets are presented, focusing on their mathematical structures and multidimensional applications.
5. **Advanced Extensions of Hyperneutrosophic Sets – Dynamic, Quadripartitioned, Pentapartitioned, Heptapartitioned, and m-Polar**: This chapter introduces advanced frameworks for solving highly nuanced problems in uncertainty modeling.
6. **Advanced Extensions of Hyperneutrosophic Sets – Cubic, Trapezoidal, q-Rung Orthopair, Overset, Underset, and Offset**: Building on foundational Neutrosophic theory, this chapter extends its application to cubic and trapezoidal constructs and explores their mathematical implications.
7. **Specialized Classes of Hyperneutrosophic Sets – Support, Paraconsistent, and Faillibilist Sets**: Unique classes of Neutrosophic Sets are analyzed, addressing theoretical challenges and opening new avenues for exploration.
8. **MultiNeutrosophic Sets and Refined Neutrosophic Sets**: This chapter highlights two advanced frameworks and examines their adaptability to multidimensional uncertainty.
9. **Advanced Hyperneutrosophic Set Types – Type-m, Nonstationary, Subset-Valued, and Complex Refined**: Expanding the Neutrosophic framework, this chapter introduces Type-m, Nonstationary, and Subset-Valued extensions.
10. **Hyperfuzzy Hypersoft Sets and Hyperneutrosophic Hypersoft Sets**: Principles of Fuzzy, Neutrosophic, and Soft Sets are integrated with hyperstructures to introduce innovative uncertainty management tools.
11. **A Review of SuperFuzzy, SuperNeutrosophic, and SuperPlithogenic Sets**: This chapter consolidates and extends research on hierarchical uncertainty models, particularly focusing on quadripartitioned, pentapartitioned, and heptapartitioned forms.
12. **Advanced SuperHypersoft and TreeSoft Sets**: The chapter presents six novel concepts, enhancing the flexibility of soft set-based models for decision-making.
13. **Hierarchical Uncertainty in Fuzzy, Neutrosophic, and Plithogenic Sets**: The final chapter surveys hierarchical uncertainty frameworks and their advanced extensions, bridging multiple set-theoretic concepts for modeling complex systems.

ISBN 978-1-59973-817-8

9 781599 738178 >